Optimal Correction of the Public Debt and Measures of Fiscal Soundness*

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Abstract

This paper derives the optimal response of the primary budget balance to changes in the public debt as a share of gross domestic product (GDP) in a stochastic model of debt. Under the optimal solution, the surplus reactivity to the debt-GDP ratio is independent of the debt ratio itself, but its size depends on the degree of uncertainty surrounding the impact of fiscal policies. We characterize the properties of the optimal control policy by proposing different metrics that may be used to assess fiscal soundness and as early warning indicators of fiscal imbalances.
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1 Introduction

In the last decade and a half since the Great Recession and the global financial crisis, dramatic fiscal developments have brought renewed attention to the issue of public debt sustainability. The economic crisis triggered by the SARS-CoV-2 outbreak has brought about a sharp increase in public debt in most countries. Since the onset of the health emergency public debt-GDP ratios have been soaring to levels never seen in any year during and after the 2008-2009 crisis. The extreme economic slowdown is requiring an unprecedented fiscal stimulus, although at the end of this very unusual period sustainability issues may emerge heavily, especially when financial markets will start again to price the risk of default. Nonetheless, the expensive legacy of the pandemic will push governments to find the right path between fiscal support to ward off a second economic downturn, and proper restraints to safeguard fiscal solvency. This issue has turned out to be of particular concern, especially for heavily indebted countries of the euro area, where the risk of a debt overhang has already kicked in after the Global Financial Crisis. This is why the issues of public debt control and of sustainability analysis are gaining momentum once again.[1]

In this respect, the exceptionally large uncertainty surrounding economic developments, including the effects of fiscal actions and the evolution of public debt, calls for the development of new tools of analysis that allow us to study the issue of the debt-GDP ratio control in a stochastic environment and that provide measures for fiscal solvency and stability that may come in handy as early warning indicators in the surveillance of fiscal imbalances and as indicators of the soundness of consolidation packages. To this end this paper proposes a continuous-time stochastic model of debt where a budget rule automatically triggers a correction mechanism of the primary balance to the debt ratio. The problem is that of a government wishing to cut down the current level of the debt ratio, while uncertainty comes through shocks that may frustrate or magnify the effects of the fiscal rule itself, given the existence of interdependencies between the fiscal stance and other determinants of debt accumulation, such as interest rates and economic growth.

By relying on optimal control theory and applying the Hamilton-Jacobi-Bellman equation, we show that under the optimal Markov control of the relationship between the primary balance and the debt ratio is linear.[2] The optimal reactivity of the primary balance to the debt ratio is decreasing in the degree of uncertainty surrounding the effects of fiscal policies. Thus the optimal correction rule prescribes a more vigorous response when the impact of fiscal consolidation is less uncertain.

Given the optimal control, we characterize the properties of the solution and derive a first measure of 'safe' public debt, that is the debt level that a government is expected to stabilize under most circumstances. We then propose different

[1]See Blanchard et al. (2021) who discuss the urgent need to redesign EU fiscal rules.

[2]For a study in which the objective of the government is, instead, that of keeping the level of output closer to a reference value in the attempt of stabilizing the economy over the business cycle, see Correani et al. (2014), who use optimal control theory and apply the Hamilton-Jacobi-Bellman equation in a stochastic IS-LM model.

metrics that can be used to assess the soundness of a consolidation plan and as early warning indicators of fiscal imbalances in the presence of uncertainty. This is a useful exercise to single out the role of uncertainty in setting the optimal control and identify the major factors that may undermine the achievement of a fiscal target.

One of the measures we propose simply refers to the time needed to reach an expected debt ratio. Clearly, the longer the time interval necessary to reduce the debt ratio by a given amount, the lower the soundness of the fiscal policy put in place. Nonetheless, the time necessary to reach a targeted amount will be affected by GDP growth, the rate of return on debt, and the uncertainty surrounding the effects of fiscal policy. A second measure is instead based on the probability to reach a fiscal consolidation objective in a given time interval. The third measure we propose is the probability that the debt ratio, in case of its exit from a given interval as a result of shocks, will not depart 'too much' from it. The idea is to gauge the fiscal soundness in the case that adverse shocks are able to push the debt ratio out from a given interval. In this context, a high probability would reflect good resilience to outside shocks, and so would be a sign of fiscal soundness. Conversely, favourable shocks may push the debt ratio in the right' direction, so accommodating fiscal consolidation. This measure relies on the concept of harmonic measure, describing the probability distribution of the debt ratio as it hits the boundary of a given open interval.[3] As far as we know we are the first to employ an approach based on a harmonic measure to construct a measure of fiscal resilience. Hinging on this metric we are also able to derive a second measure for the safe level of debt. We will see that this second measure is less stringent than the former, that is why it could be interpreted as a minimal condition to be met to have the chance of stabilizing debt.

We argue that the all the three proposed measures may be fruitfully used as indicators of the goodness of a fiscal package and as pre-alert indicators of fiscal imbalances. In particular, the first two measures are more appropriate to evaluate the health of public finances in a medium- and long-run perspective, while the third measure is more appropriate to assess the government's ability to service all the upcoming obligations in a short-run perspective.

This paper is related to the vast literature on debt sustainability assessment and on measures of fiscal soundness. In the last decades, this literature has evolved to account for the fact that debt sustainability analysis requires awareness of the uncertainty surrounding the evolution of public debt.[4] This strand of literature, mostly developed at the institutional level and within international organizations, explicitly accounts for the fact that fiscal solvency and debt behaviour depend on the future dynamics of economic fundamentals that are not known for sure and that may be extremely volatile (e.g. Berti 2013, Rozenov 2017 and Cherif and

[3]The theory of harmonic measure has been extensively used in several applications, such as the corona problem and in mapping problems. It has particularly interesting applications in probability theory, especially in relation to Brownian motions. See Garnett and Marshall (2005) for a survey of the theory and applications concerning this measure.

[4]For recent comprehensive overviews of the tools commonly used to measure public debt sustainability and anticipate fiscal vulnerabilities, see Corsetti (2018) and Debrun et al. (2019).

Hasanov 2018), highlights the importance of designing fiscal rules that are truly operational (see Eyraud et al. 2018), and proposes methods to quantify the fiscal stress (e.g. Balducci et al. 2011 and Pamies Sumner and Berti 2017) and the fiscal space (e.g. Ghosh et al. 2013).

From a methodological point of view, the closest predecessors of our paper are those dealing with stochastic control problems of the debt-GDP ratio. In particular, Ferrari (2018) explores the case of a government whose objective is that of reducing the debt ratio through the minimization of two opposing costs, namely the expected opportunity cost of having debt on the one hand, and the expected cost from the reduction policy on the other hand. In more detail, Ferrari (2018) shows that the solution of the control problem is related to that of an auxiliary optimal stopping problem. Put it differently, dealing with the optimal stopping problem is equivalent to working out the solution to the corresponding control problem. In conclusion, the optimal policy is found to be that of keeping the debt ratio under an inflation-dependent ceiling. Ferrari and Rodosthenous (2018) introduce the problem of a government managing the debt ratio in a stochastic continuous-time model where uncertainty comes through a macroeconomic risk process affecting the interest rate bearing on public debt. The exogenous risk process is modelled as N-state continuous-time Markov chain, while the government faces a trade-off between the potential benefits from high public investments and the costs deriving from having an excessive debt ratio and austerity policies. At the optimum, the government would keep the debt ratio in an interval whose boundaries depend on the possible states of the Markov process. Callegaro et al. (2020) study the problem of a government aiming at reducing the debt ratio under partial information where the underlying macroeconomic conditions are not directly observed. Cadenillas and Huamán-Aguilar (2016) develop a stochastic debt control model to find the optimal ceiling for the government debt. As in Ferrari (2018) the government objective is that of minimising the trade-off between the opportunity cost of having debt and the cost from arising from its reduction. Cadenillas and Huamán-Aguilar (2016) obtain a closed-form solution for the optimal government debt ceiling and find that the fiscal policy will be active if the debt ratio is greater than the optimal debt ceiling, while a passive fiscal policy will be desirable if the debt is lower than the ceiling.[5] In a subsequent paper Cadenillas and Huamán-Aguilar (2018) study the optimal debt ceiling accounting for the fact that the ability of the government to reduce its debt ratio is bounded. Recently, Lindgren (2021) has proposed a stochastic optimal control model of public debt dynamics, where the interest-growth differential is stochastic, as in the present paper, but the fiscal authorities control the evolution of the primary balance variable without conditioning its evolution to a linear rule. The model is then extended to embody feedback effects of debt and fiscal stance on the interest-growth differentials, assuming that policymakers do not directly observe these effects.

[5] In a previous contribution Huamán-Aguilar and Cadenillas (2015) propose a stochastic model for government debt control under the assumption that debt may also be issued in foreign currency. They show that for high debt aversion and exchange rate uncertainty, it is optimal to reduce the share of the debt burden denominated in foreign currency in favour of domestic currency.

The remainder of the paper is structured as follows. Section 2 lays out the model. Section 3 introduces and solves the optimisation problem of the government. Section 4 presents different measures of fiscal soundness under the optimal control policy. Section 5 presents concluding remarks. All the technical details are relegated to appendixes.

2 The Model Setup

A simple starting point for the formal discussion of public finances is the flow budget constraint of the government which dictates that the next period debt is given by the current period debt times a gross interest factor minus the primary balance (government revenues minus expenditures excluding interest payments). This relationship can be easily written in terms of GDP share as follows:

$$X_{t+1} = \frac{1+r_t}{1+g_t} X_t - S_t, \tag{1}$$

where X is the the stock of public debt as a proportion of GDP, S is the primary balance as a share of GDP, r denotes the interest rate on government debt, and g is the growth rate of GDP.

By making use of the approximation $\frac{1+r}{1+g} \approx 1 + r - g$, equation (1) can be equivalently expressed as follows:

$$X_{t+1} = X_t(1+\alpha_t) - S_t. \tag{2}$$

where $\alpha_t \equiv r_t - g_t$.

We assume that the behaviour of the primary balance is described by a debt-based reaction rule.[6] In particular, we focus on a reaction rule to the debt ratio of the form:

$$S_t = \rho_t X_t, \tag{3}$$

where ρ_t is the government control variable measuring the strength of the primary surplus response to the debt ratio X_t. The above rule simply prescribes that the primary surplus is a function of the debt ratio. A positive response of the primary surplus implies that a government is taking corrective actions that counteract the changes in debt, consistently with empirical evidence.[7] The implicit idea here is that as long as the primary balance reacts sufficiently to debt, any debt is sustainable. However, in practice, there is a limit for the maximum magnitude of primary balance adjustment that would be plausible. For instance, at very high debt levels the corrective measures would require a degree of fiscal austerity that would be socially unbearable.

[6] This rule is consistent with the spirit of the EU fiscal rules for national fiscal policies, but it does not mean capturing the complexity of the EU fiscal framework.

[7] There are several studies investigating the positive relationship between the primary balance and public debt. See e.g. Bohn (1998), Medeiros (2012), Lukkezen and Rojas-Romagosa (2013), Bartoletto et al. (2014), Everaert and Jansen (2018).

Given (3), equation (2) can be re-written as:

$$X_{t+1} - X_t = (\alpha_t - \rho_t)\, X_t, \tag{4}$$

Given the uncertain feedback effects of the fiscal adjustments on economic variables, α_t is assumed to have two components, a deterministic component capturing long-run fundamentals, and a random factor that depends on the degree of the fiscal effort ρ_t. In particular, we assume that

$$\alpha_t = \alpha - \sigma \rho_t \epsilon_t, \tag{5}$$

where the term α reflects the long-term interest-growth differential, σ represents the diffusion coefficient meant to transmit uncertainty to the response action of policymakers and ϵ_t is the white noise in discrete time. The analysis is conducted under the assumption that $\alpha > 0$, that is to say that on average the interest rate on public debt is higher than the GDP growth rate (condition for dynamic efficiency). It implies that the higher the interest-growth differential, the larger the fiscal effort necessary to stabilize the debt ratio or to reach a reduction target. This assumption is consistent with the empirical evidence for most of the advanced economies in 'normal times' (away from crisis). More specifically, using data for the period 1999-2008, Escolano et al. (2017), show that the interest rate-growth differential is on average positive for most of the advanced economies, but negative for many non-advanced economies undergoing a catch-up process. Our analysis is thus conducted having in mind a mature economy, although we recognize that advanced economies may also experience prolonged periods of negative interest-growth differential, especially after the Global Financial Crisis, as documented by Escolano et al. (2017) and as discussed by Blanchard (2019).[8] Under the alternative assumption, $\alpha < 0$, the public debt on average would follow a stable process and the debt ratio would tend to decline. This would be possible also in the presence of persistent primary deficits, since in these circumstances borrowing would pay for itself.

The second term in (5) captures the uncertainty that may surround the final outcome of any fiscal intervention and is related to the macroeconomic effects of fiscal policy on the interest-growth differential.[9] Clearly, this stochastic component affects the effective size of the primary balance ratio. This uncertainty may originate from several factors and macroeconomic feedback effects.[10] An ambitious fiscal consolidation plan may deteriorate economic conditions to such an extent that tax revenues decline and social spending increases, partially frustrating the initial correction. This self-defeating mechanism of the corrective measure may

[8]More recently, Zhou and Mauro (2020), using historical, show that negative interest-growth differentials have frequently occurred in both advanced and emerging economies, and have often displayed a persistent pattern.

[9]It should be noted that, contrary to what assumed in Lindgren (2021), here policymakers directly observe these potential effects of fiscal policy decisions on the interest-growth differential

[10]See also Balibek and Köksalan (2010) for a model of debt management taking into account the uncertainty concerning the future state of the economy.

thus lead to a negative shock.[11] Similarly, a strong corrective fiscal intervention may undermine growth prospects, pushing private investors to cut down their investment plans, leading to a knock-on effect on the level of economic activity and thus to the debt-GDP ratio. Different beliefs about the type of fiscal consolidation may give rise to waves of optimism that may improve the performance of the consolidation itself or, alternatively, to waves pessimism that may magnify the contractionary effects of the ongoing specific fiscal plan.[12] However, debt reduction may be also conducive to positive shocks. Indeed, a surplus correction may increase the confidence of private investors, boosting market confidence and lowering risk premia in countries with high debt, so that we may observe a positive effect on the surplus ratio. Further, possible non-Keynesian effects of fiscal policy may give rise to beneficial effects on the budget balance by magnifying the effects of restrictive fiscal intervention.

A credible fiscal consolidation plan can signal future reductions of distortionary taxes and, therefore, an increase in permanent income, producing an increase in private consumption. Private investments may also respond positively, via the interest rate channel or an expected lower tax burden in the future.[13] Non-Keynesian effects of fiscal policy could then account for positive shocks. According to the empirical evidence discussed in Alesina et al. (2018), fiscal corrective measures based upon spending adjustments are much less costly in terms of output losses than those based upon tax adjustments. In general, the stochastic component in (5) is meant to capture the dilemma faced by fiscal authorities that must strike a balance between the need of a strong action and the uncertainty that the action itself may magnify.

To analyze how the debt accumulation evolves in the presence of a stochastic term, we substitute the fiscal rule (5) into equation (4) and obtain the following expression in discrete time

$$X_{t+1} - X_t = (\alpha - \rho_t)\, X_t dt - \sigma \rho_t X_t \epsilon_t \,, \tag{6}$$

which in continuous time becomes:

$$dX_t = (\alpha - \rho_t)\, X_t dt - \sigma \rho_t X_t dW_t \,, \tag{7}$$

where W_t is an one-dimensional Brownian motion with zero mean and density function given by a Gaussian exponential law of the type:

$$W_t \;\sim\; \frac{e^{-\frac{y^2}{2t}}}{\sqrt{2\pi t}} \tag{8}$$

[11]See e.g. DeLong et al. (2012). According to empirical evidence, fiscal multipliers are large during recessions and small when the economy operates close to potential. See Auerbach and Gorodnichenko (2012) and Corsetti et al. (2013). On the positive effects of fiscal expansion, see e.g. Blanchard and Perotti (2002).

[12]The effects of fiscal actions also depend on the underlying monetary-fiscal policy regime, on expectations about future regimes and on the credibility of an announced fiscal plan. All these factors are not directly controlled by policymakers. In that matter, for a comprehensive discussion on how “darned hard” fiscal analysis is, see Leeper (2015).

[13]See e.g. Giavazzi and Pagano (1990) and Alesina and Ardagna (2013), and the discussion in Padoan et al. (2013).

where we have identified the term $\epsilon_t \, dt$ in (6) with dW_t in (7).

The variable described by equation (7) is an Itô process with a unique solution, since it satisfies the two conditions for the existence and uniqueness of the solution.

For more details about these conditions and the mathematical features underlying equation (7), see Øksendal (2003).

Before proceeding with our analysis we must acknowledge that the assumption regarding the Brownian nature of W_t allows us to have a well-defined problem of stochastic calculus. However, empirical evidence points to the fact that very often the interest-growth rate differential may change abruptly, as a result of a sudden change in market sentiments or in investors' expectations regarding the fiscal risk of an economy. These phenomena may also persist for a certain time span. See e.g. Zhou and Mauro (2020). Clearly, with our assumption on the nature of W_t we are able to capture sudden shifts in the interest-growth differential, but we cannot capture its persistent deviations from the mean.

3 The Optimisation Problem

In this section we consider the problem of a government aiming at reducing the current level of the debt ratio. Moreover, the fiscal authority is assumed to have always access to the available policy tool ρ, which is the strength of the primary balance response to the debt ratio. The government is assumed to be increasingly worse off the larger the debt ratio. The idea is that the government faces an instantaneous loss related to the rising public debt. Notably, a large public debt may crowd out private investment undermining growth prospects.[14] In addition, one of the potential effects associated with excessive public debt is that of a major increasing perceived risk that a country may default on its debt. This change in market sentiments may push an economy towards a bad equilibrium through self-fulfilling upward effects on yields, and thus debt may become unsustainable. Moreover, since the unpleasant arithmetic of Sargent and Wallace (1981) it has been well known that it is impossible for monetary authorities to sustain low inflation in the presence of excessive public debt and profligate fiscal policy. Finally, the implementation of restrictive fiscal policies in response to an increase in the debt ratio may hinder growth, especially during a recession (see DeLong et al. 2012).

This assumption translates in a quadratic expected loss function J_T of the type:

$$J_T = \mathbf{E}\left[(X_T)^2\right] \equiv \int_\Omega (X_T)^2 \, d\mathbb{P}, \tag{9}$$

where X_T is the stochastic level of debt at time T and $\mathbf{E}$ denotes the expectation value with respect to the probability law of X, that is with respect to the probability measure $\mathbb{P}$.

[14] There is a quite vast empirical literature which shows that there is a negative correlation between public debt and economic growth (see e.g. Reinhart and Rogoff 2010, Woo and Kumar 2015). Yet, the casual interpretation of the correlation is an open issue since there might be cases in which causation goes from low growth to high debt, rather than the other way round.

The time T is the *exit time* of the process X_t from its interval $(\underline{x}, \overline{x})$, that is it yields

$$T = \inf_t \{t > 0 \ \text{ such that } X_t \leqslant \underline{x}\}, \tag{10}$$

with $\mathbf{E}[T] < \infty$.

As usual in the dynamic programming literature, we now let the controlled diffusion X start at time s from level $x > 0$, that is

$$\begin{cases} dX_t^{s,x} = (\alpha - \rho_{t-s}) X_{t-s}\, dt - \sigma \rho_{t-s} X_{t-s} dW_{t-s}, \\ \text{sub} \qquad X_s^{s,x} = x. \end{cases} \tag{11}$$

The optimization problem now reads:

$$\phi(s, x) = \inf_{\rho \in \mathcal{A}} \mathbf{E}\left[(X_T^{s,x})^2\right], \tag{12}$$

where $\phi(s, x)$ denotes the value function and $\mathcal{A}$ the family of admissible controls. To solve the system, we use the Hamilton-Jacobi-Bellman (HJB) equation.[15] In Appendix B it is shown that the optimal Markov control $\hat{\rho}(t, X)$ is

$$\hat{\rho}_t = \hat{\rho}(t, X_t(\omega)) = \hat{\rho} = \frac{1}{\sigma^2}. \tag{13}$$

From the above result, two remarks are in order. First, the optimal correction factor is independent of the debt ratio. This implies that at the optimum the relationship between the surplus ratio and the debt-output ratio will be linear. This is because the assumed functional form for our objective function is only based on the distance of the debt from its target value and does not depend on other costs (e.g., the fiscal effort). See Øksendal (2003, chap. 11) for more details.

Second, when the coefficient diffusing uncertainty is high, the correction factor should correspondingly be low. The idea behind this result is that large shocks can potentially undermine or magnify the effectiveness of the fiscal effort so that it is 'optimal' to limit the magnitude of the correction mechanism itself.[16]

If we insert the optimal control (13) into the evolution equation in (7), this equation becomes

$$dX_t = \left(\alpha - \frac{1}{\sigma^2}\right) X_t\, dt - \frac{1}{\sigma}\, X_t\, dW_t\,, \tag{14}$$

whose solution, by virtue of Itô's lemma, is

$$X_t = x\, e^{\left(\alpha - \frac{3}{2\sigma^2}\right)t} e^{-\frac{1}{\sigma} W_t}. \tag{15}$$

From equation (15) we observe how under the optimal rule the time path of the debt ratio may be affected by uncertainty. The stochastic component $e^{-\frac{1}{\sigma}W_t}$ adds to a deterministic factor driving down the debt ratio. In this framework, it is informative to consider what this non-stochastic fiscal factor, based on the

[15] For the existence of a unique solution to this problem, see Øksendal (2003). For more details about the HJB methodology, see Fleming and Soner (2006) and Stengel (1986).

[16] To find an explicit value for the value function, see Appendix C.

underlying debt dynamics equation, might tell us about the fiscal effort needed to achieve a given target debt-to-GDP ratio.[17] On top of this, we can also consider how the introduction of the stochastic component may lead to following different trajectories with some uncertainty. Besides, it is worth noticing that the optimal consolidation effort reached within a certain time period may not be realistic because fiscal austerity beyond a certain level and given time frame may not be socially acceptable. In this respect, to depict a realistic dynamics of (15) and to bring the model to data (adapting the model parametrization to country-specific experience) we consider a fiscal effort compatible with the primary fiscal surpluses observed in Belgium.[18]

In Figure 1 we plot the deterministic component of (15) (dashed line) and five sample paths for the debt ratio evolving according to equation (15) using AMECO data for Belgium for the period 1995-2019. Specifically, we set the initial value of the debt to 131.3%, that is Belgium's observed debt ratio in 1995, α to 0.0066, that is the observed mean value for $r - g$, and ρ to 2.7%, that is consistent with the observed mean value for the primary balance ratio over the period 1995-2019. The value of σ is then implied from the optimal solution (13). We observe how that debt ratio tends to decline over time in the deterministic setup, as prescribed by the optimal control policy. This can be interpreted as the underlying trend of the debt dynamics. As expected, in presence of the Brownian motion we observe some temporary and persistent fluctuations away from this trend. Moreover, the time needed to reach the fiscal consolidation objective varies considerably across different paths. Not surprisingly, the deterministic time path reflects the downward speed of the debt ratio reduction observed in Belgium over 1995-2019.

However, we observe that in some particularly adverse circumstances, the public debt ratio may dramatically increase, thus requiring a stronger line of fiscal manoeuvres that may turn out to be politically and economically unsustainable. This leads us to the notion of 'debt limit' which is a level of the debt ratio above which keeping its level under control would require impossibly large primary surpluses, and to the notion of 'safe debt' that is the initial level of debt ratio that is expected to be reduced or stabilized over a given time frame, even under severely adverse conditions. Adapting to our framework the methodology proposed by Debrun et al. (2020), one can fix a debt limit at 170% of GDP, set the tolerated probability of exceeding this limit over a 5-year time horizon to 0.05, and then compute the level of the debt ratio, x, that, according to (13), satisfies the following condition:

$$\mathcal{P}(X_t > 170) = \mathcal{P}\left[x^{safe}\, e^{\left(\alpha - 3/(2\sigma^2)\right)t} e^{-W_t/\sigma} > 170\right] = 0.05, \tag{16}$$

where x^{safe} is our measure for the safe debt.

[17] The IMF practice and metrics concerning the fiscal risk facing an individual country are also based on such a deterministic framework. See Escolano et al. (2017).

[18] Belgium's strong record of large primary fiscal surpluses and of progress in reducing high public debt has been remarkable over the last 25 years. In this time frame, Belgium's fiscal effort is to be considered the upper bound of a socially-acceptable long-term fiscal consolidation plan in Europe.

This debt ratio can then be used as a measure of safe debt. Table 1 reports our simulation results for different values of fundamentals and uncertainty. Clearly, the safe debt is decreasing in α: weak fundamentals, for a given level of uncertainty, require a lower safe debt. On the other hand, more uncertainty, which can also favour consolidation, allows for higher safe debt.

Now, given the presence of uncertainty, the following questions naturally arise. What is the time needed to meet a fiscal target? How robust is the adjustment rule to adverse shocks? Or better, what is the probability that given the materialization of adverse shocks public debt is still on the right track towards a preset fiscal goal? In the next section, we will address these questions by proposing different approaches.

4 Measures of Fiscal Soundness

In this section, we assess the properties of the debt dynamics under the optimal policy (13) proposing different measures of fiscal soundness. Specifically, we will first look at the time necessary to reach an expected fiscal consolidation objective; then we will compute the probability of reaching a fiscal objective in a specific time horizon. These two measures provide us with different information about the ability of the government to meet its fiscal consolidation target in a stochastic environment from a medium- and long-run perspective. In particular, the first measure pinpoints the time needed to reach a given objective that is expected to be achieved in the presence of shocks. This may be seen as the time resistance towards the objective when the system is placed under pressure. The second measure refers to the probability of reaching a fiscal consolidation target in a given time horizon. A higher probability is clearly a sign of a sound fiscal policy.

Finally, we will propose a measure of fiscal soundness based on the probability that the debt ratio, departing from a given interval because of shocks, tends to decline. Thus, a high probability conditional to the exit of the debt ratio of a predetermined interval may be interpreted as the confidence to absorb adverse shocks that may undermine fiscal stability. From this point of view, this measure of fiscal resilience may be used to assess the health of public finances in the short run.

4.1 Time Needed to Reduce the Debt Ratio

What is the time needed to reduce the debt ratio by a given amount? How do uncertainty and fundamentals affect the time required to meet a given target? These questions are relevant issues for policymakers since the time dimension represents an important facet of the evaluation of a fiscal consolidation process. In particular, adverse cyclical factors and changing political conditions may considerably expand the time eventually needed to reach a given objective, prolonging the time span of policy action. The credibility of any fiscal reform also depends on the expected time necessary to reach an established aim. The more distant in the future the achievement of the final goal, the less credible the policy action will be.

From (15), recalling (8), the expected value of the debt ratio $E[X_t]$ is

$$\mathbf{E}[X_t] = x\, e^{\left(\alpha - \frac{3}{2\sigma^2}\right)t} \int_{\mathbb{R}} e^{-w/\sigma} \frac{e^{-w^2/(2t)}}{\sqrt{2\pi t}}\, dw = x\, e^{\left(\alpha - 1/\sigma^2\right)t}, \tag{17}$$

while the variance is

$$\mathbf{E}[X_t - \mathbf{E}[X_t]]^2 = x^2\, (e^{\frac{1}{\sigma^2}t} - 1) e^{2\left(\alpha - \frac{1}{\sigma^2}\right)t}. \tag{18}$$

From (17) the expected value of the debt ratio declines over time provided that condition $1/\sigma^2 > \alpha$ holds. The speed of convergence towards an expected target is clearly increasing in α and in σ. If $1/\sigma^2 > \alpha$, then the variance of X_t will display hump-shaped dynamics over time.

To better illustrate the behaviour of moments as time changes we will make use of a numerical example. Figure 2 presents the expected value of the debt ratio $E(X_t)$ and its standard deviation σ_{X_t}, given an initial debt ratio $x = 130\%$, $\alpha = 0.01$ under three different values of σ. Higher uncertainty will expand the time necessary to reach the objective as a result of the fact that the government will find it 'optimal' to slow down the fiscal effort in response to the higher unpredictability of the final outcome of the policy intervention. As an example, after 10 years for $\sigma = 5$, $E(X_t)$ will be about 20 p.p. lower than what observed under higher uncertainty, that is for $\sigma = 7$. After 40 years $E(X_t)$ is close to 40% with $\sigma = 5$, while for $\sigma = 7$ the expected value is still more than 30 p.p. above. However, a strong reaction to the debt ratio initially generates a high variability especially when σ is lower. This is because the optimal rule (13) prescribes a strong reaction to the debt ratio when σ is low, so inducing a substantial feedback effect on α_t (see eq. 5). At later stages, instead, the standard deviation declines faster the lower the degree of uncertainty. As long as the debt ratio declines, the amount of uncertainty is sharply reduced. This is the result of the initial trade-off faced by the policymaker at the earlier stages of the adjustment towards targeted debt reduction, discussed in Section 2.

Figure 3 shows the role played by market fundamentals in determining the time path of the expected value of the debt ratio and of its standard deviation. We observe that, other things being equal, the higher α, reflecting weak economic fundamentals, such as structural low growth and/or high-interest rate, the slower the convergence towards the objective, and thus more the time needed to meet the established objective, and the higher the variability. Hence, as expected, a high α severely reduces the stabilizing properties of the rule. Overall, the effects of changes in market fundamentals are magnified in the presence of high uncertainty. This can be easily explained by close inspection of equation (14), where for an increase in σ the role of market fundamentals becomes pivotal in shaping the time path of the debt ratio. In the presence of high uncertainty the optimal rule, in fact, implies a weaker reaction, so that the adjustment of the debt ratio towards a given target relies on market fundamentals to a greater extent.

Table 2 summarizes the above findings presenting the time needed to reach different debt ratios that are expected to be achieved in the presence of shocks for different values of σ and α. We consider four different expected fiscal consolidation goals and from (17) we compute the time needed to reach the objective. As before, the initial value x is set to 130%. In the more favourable scenario, with α fixed at 0.007 and σ at 5, the time needed to reduce the debt by 20 p.p. is 5 years, while in the worst scenario, with α at 0.015 and σ at 7, is 31 years. Similarly, reducing the debt by 40 p.p. is feasible in 11 years under favourable conditions and in 68 years under adverse circumstances. It should be noted that for low uncertainty the number of years necessary to reduce the debt depends non-linearly on the size of α. For low uncertainty, in fact, the optimal rule dictates a stronger adjustment.

4.2 Probability of Reaching a Fiscal Objective

We now show how uncertainty and fundamentals affect the probability of reaching a fiscal target. Since the debt ratio is constantly bounced around by a number of shocks, the reduction of debt towards a target is uncertain. Indeed, debt trajectories are surrounded by uncertainty, as a result of two opposing forces, the correction rule pushing debt down on the one hand, and adverse shocks that may drive debt up on the other hand. Therefore, in a stochastic environment, it becomes relevant for policymakers to measure the degree of confidence associated with the effectiveness of the fiscal action at play.

To this end we use (15) to compute the probability that the debt ratio reaches specific targets, say z, after a given time interval:

$$\mathcal{P}(X_t < z) = \mathcal{P}\left[x\, e^{\left(\alpha - 3/(2\sigma^2)\right)t} e^{-W_t/\sigma} < z\right]. \tag{19}$$

In our numerical examples, we compute the probability that the debt ratio reaches specific targets after 5, 10 and 20 years under different parametrizations for σ and α. The initial value of debt ratio x is set to 130%. By using the probability law (19) induced by the process X_t, we can calculate the probability that the public debt is lower than some fixed values, as it is summarized in Table 3. As expected, adverse fundamentals (i.e. high values for α) and high uncertainty undermine fiscal sustainability.

As an example, the probability of closing the debt by 20 p.p. in 10 years (namely the case $X < 110$), corresponding to an average yearly reduction by 2 p.p. over 10 years, is around 72% under favourable GDP-growth-interest-rate conditions ($\alpha = 0.007$) and in the case that the coefficient diffusing uncertainty is low ($\sigma = 5$). In this situation, the stabilising effect of the optimal rule tends to prevail over the adverse feedback effects on the interest-growth differential, making this objective confidently reachable. As the economic fundamentals weaken (that is α increases) the corresponding probability to meet this target falls, highlighting how the outcome of the fiscal consolidation at stake is affected by long-term/structural components of the economic system. We observe that this measure of probability drops dramatically as σ increases. In this case, there are two concurring forces leaning against the reaching of a fiscal target. On the one hand, a high σ entails a lower fiscal effort so as to drag down the probability of meeting the target; on the other hand, a high σ increases the uncertainty of the fiscal intervention. As a result, the probability of closing the debt by 20 p.p. in 10 years when $\sigma = 7$ is now around 49% in the worst-case scenario ($\alpha = 0.015$).

4.3 Fiscal Resilience Measure

In this section, we propose a fiscal resilience measure aiming at assessing the health of public finances in a stochastic environment from a short-run perspective. In detail, we assume that the government is committed to following the optimal rule (13). However, the stochastic component W_t in (5) may push the debt ratio up for some time, thus frustrating any policy effort and undermining fiscal stability. In this context, we need to further operationalise the concept of fiscal resilience by choosing an appropriate indicator able to identify emerging risks.

To this end we assume that at time 0 the debt ratio x is between two extremes, say $\underline{x}$ and $\overline{x}$, that is $x \in D := (\underline{x}, \overline{x})$. These two extremes may reflect the boundaries within which we may expect the debt ratio to vary in the short run. In a process of fiscal consolidation x will sooner or later exit this interval from below, however, adverse shocks may push x to leave this interval from above at least in the short term.

We then address the following questions. What is the probability that, given the uncertainty, at time t the debt ratio departing from this interval is as close as possible to $\underline{x}$? What is the role played by fundamentals and uncertainty?

In order to construct this probability we make use of the concept of harmonic measure. Formally, a harmonic measure of X_t describes its distribution as X_t hits the boundaries of D, namely $\underline{x}$ or $\overline{x}$. By virtue of the diffusion theory related to the Itô stochastic processes, we can associate the following second-order ordinary differential equation to equation (14) as follows:

$$\frac{x^2}{2\sigma^2} f''(x) + \left(\alpha - \frac{1}{\sigma^2}\right) x f'(x) = 0. \tag{20}$$

Let $f \in C^2(R)$ be a solution of this differential equation. Also, let $(\underline{x}, \overline{x}) \subset R$ be an open interval such that $x \in (\underline{x}, \overline{x})$ and put

$$\tau = \inf \left\{ t > 0 \; : \; X_t \notin (\underline{x}, \overline{x}) \right\}, \tag{21}$$

where τ measures the first instant of time in which the debt ratio does not belong to the interval $(\underline{x}, \overline{x})$.[19]

Recalling the Dynkin's formula it is possible to give a formal expression for the probability that debt is bending towards the lower bound $\underline{x}$. If $f(\overline{x}) \neq f(\underline{x})$, by using the Dynkin's formula then the probability may be written as

$$\mu^x(\underline{x}) = \frac{f(\overline{x}) - f(x)}{f(\overline{x}) - f(\underline{x})}, \tag{22}$$

[19]We are assuming that $\tau < \infty$ almost sure with respect to the probability law Q^x by means of the Brownian motion.

that is the harmonic measure μ of X on $\underline{x}$.[20] For a formal proof of how (22) is obtained, see Appendix D. Such a harmonic measure is the probability that, in the first instant of time τ in which the process X_τ does not belong to the fixed open interval $(\overline{x}, \underline{x})$, the process assumes the value $\underline{x}$. At the operational level this interval can be anchored to the deficit limit given by the available fiscal space or set institutionally, as the deficit limit enshrined in the Maastricht Treaty.

From an economic point of view, the harmonic measure (22) may have a twofold interpretation. On the one hand, it may be seen as an index of risk about how much a system is vulnerable to potential adverse shocks. In our setup, a state jump into the wrong direction $\overline{x}$ may represent an early warning indicator of fiscal imbalances. On the other hand, $\mu^x(\underline{x})$ may be portrayed as the government's ability to bend towards a target for a given α and σ, under the optimal correction rule and in the presence of external perturbations. The first interpretation highlights the capability to withstand negative shocks, while the second one refers to the capability of meeting an objective. Nevertheless, both interpretations outline the idea of resistance against adverse conditions. This is why in terms of fiscal resilience the two interpretations are interchangeable.

After some manipulations (see Appendix E for more details), it is possible to give an explicit expression for (22) as follows:

$$\mu^x(\underline{x}) = \frac{\overline{x}^{3-2\sigma^2\alpha} - x^{3-2\sigma^2\alpha}}{\overline{x}^{3-2\sigma^2\alpha} - \underline{x}^{3-2\sigma^2\alpha}}. \tag{23}$$

The value of $\mu^x(\underline{x})$ thus depends on the specific setting of α and σ. Figure 3 shows how this probability changes for different values of α and σ, where we have set $x = 130\%$, $\underline{x} = 125\%$ and $\overline{x} = 135\%$. The horizontal dashed line corresponds to $\mu^x(\underline{x}) = 0.5$. The three plotted curves intersect the horizontal lines for values of σ, such that $1/\sigma^2 = \alpha$. We observe that a lower α (i.e. more favourable economic conditions) increases the probability that when departing from the interval $(\underline{x}, \overline{x})$ the debt ratio goes down. A higher probability would then signal a more resilient fiscal stance. A higher σ, instead, injects more fiscal policy uncertainty into the system so undermining fiscal stability. The lower probability would signal that the trajectory of the debt-GDP ratio could be significantly more worrisome than expected.

We now take a step further by using the metrics (23) to compute an alternative measure for safe public debt. The safe debt is now defined as the level of the debt ratio, x, such that, having set $\overline{x}$ to 170 (the exogenous debt limit considered in Section 2) and $\underline{x}$ to the value generated by the deterministic low of motion of debt under the optimal policy of equation (15), the probability $\mu^x(\underline{x})$ is equal to 0.95. The results are reported in Table 4 for a time horizon of 5 years consistently with the example in Table 1. This alternative measure for the safe debt has a different connotation than the one proposed in Section 2. Here the safe debt can be thought of as a minimal condition to be met so as to be on the safe side during a fiscal consolidation process. This means that in the case of exiting from the interval ($\underline{x}$,

[20] To be sure, the corresponding harmonic measure μ of X on $\overline{x}$ can be derived as $\mu^x(\overline{x}) = 1 - \mu^x(\underline{x})$.

$\overline{x}$), the probability that the debt departs from below is 0.95. Of course, a major level of uncertainty, as shown in the previous case, delivers a higher safe debt. On the other hand, contrary to the measure introduced before, here we observe that the safe debt is increasing in α. This is because a larger α requires, leaving everything else equal, higher deterministic debt and thus a narrow interval.

5 Conclusion

This paper studies the optimal debt reduction policy in a simple stochastic model of debt by using optimal control theory and applying the Hamilton-Jacobi-Bellman equation. The government is assumed to follow a simple feedback rule according to which the primary balance is adjusted to the debt-GDP ratio.

However, the government has only partial control over the primary balance, since the final impact of any fiscal intervention is surrounded by uncertainty.

In such an environment the optimal Markov control policy turns out to prescribe that the reactivity of the primary balance to the debt ratio is independent of the debt ratio itself, rather it depends only on the degree of uncertainty surrounding the effects of fiscal policies on the economy. Overall, the optimal rule envisages a strong fiscal effort under more stable economic conditions. This result suggests that a simple linear rule of primary balance adjustment to the debt ratio may be optimal, provided that the size of the adjustment coefficient is tailored to the underlying market fundamentals. We propose three different measures to assess the health of fiscal finances. The first measure is simply based on the time needed to meet an expected fiscal consolidation objective. The second measure looks at the probability of reaching a fiscal target in a given period. Clearly, these two metrics can be used to analyze fiscal soundness from a medium- and long-run perspective. The third measure, relying on harmonic measure theory, is constructed from the probability distribution of the debt ratio as it hits the boundary of a given open interval. This measure gives us the probability that the debt ratio bends toward a target and can be then interpreted as a fiscal resilience indicator that may be used to assess the stability of public finances from a short-run perspective.

As expected high growth rates and low-interest rates improve the soundness of fiscal policy, while a higher degree of uncertainty jeopardizes fiscal stability. We argue that these measures could be used as simple indicators to gauge the goodness of a fiscal consolidation plan and as an early warning indicator of fiscal imbalances.

Overall our results show that the performance of simple feedback rules, prescribing an automatic relationship between the primary balance and public debt, may be frustrated by the existence of uncertain feedback effects from the fiscal action to the interest-growth differential. These feedback effects are hard to predict and reflect the diversity of possible country-time situations that should be considered in the fiscal evaluation assessment. In this respect, we complement our results by introducing two notions of safe debt that may further help in addressing pragmatically the high-debt issue.

In this paper we have deliberately considered a parsimonious model, yet general enough to capture several dimensions of the public-debt control problem. The analysis may be extended in a number of directions. First, the analysis might be extended to account for the interaction between monetary and fiscal policies. The underlying monetary regime may facilitate or make more difficult the optimal control of public debt, and when it comes to ensuring jointly price stability and fiscal solvency, the policy trade-offs may become more severe and the optimal control problem more challenging. Second, one should consider a more sophisticated nature for uncertainty so as to capture the non-mean reverting behaviour of the interest-growth differential and the possibility of its persistently negative value. Third, the measures of fiscal resilience proposed in this paper should be compared with other fiscal indicators and their behaviour should be analyzed in practice. We leave these aspects for future research.

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The authors have no conflicts of interest to declare that are relevant to the content of this article.

Availability of data and material

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Code availability

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Tables and Figures

Table 1: Safe Debt under the Optimal Solution

	$\alpha = 0.007$	$\alpha = 0.01$	$\alpha = 0.015$
$\sigma = 5$	106.19	104.61	102.02
$\sigma = 6$	109.52	107.89	105.23
$\sigma = 7$	113.12	111.45	108.68

Note: the table reports the initial level of debt, x, such that, over a 5-year time horizon, the probability that under the optimal solution the debt ratio will exceed the debt limit of 170% of GDP is 0.05.

Table 2: Time Necessary to Meet an Expected Debt Ratio Target

σ=5			
	α=0.007	α=0.01	α=0.015
$E(X) = 110$	5	6	7
$E(X) = 90$	11	12	15
$E(X) = 80$	15	16	19
$E(X) = 60$	23	26	31
σ=6			
	α=0.007	α=0.01	α=0.015
$E(X) = 110$	8	9	13
$E(X) = 90$	18	21	29
$E(X) = 80$	23	27	38
$E(X) = 60$	37	43	61
σ=7			
	α=0.007	α=0.01	α=0.015
$E(X) = 110$	12	16	31
$E(X) = 90$	27	35	68
$E(X) = 80$	36	47	90
$E(X) = 60$	58	74	143

Note: the table reports the number of years necessary to meet an expected debt ratio target, expressed in %, for different combinations of uncertainty, σ, and fundamentals, α, given an initial debt ratio $x = 130\%$.

Table 3: Probability of Reaching a Debt Ratio Target

	10 years			20 years		
$\sigma = 5$						
	$\alpha = 0.007$	$\alpha = 0.01$	$\alpha = 0.015$	$\alpha = 0.007$	$\alpha = 0.01$	$\alpha = 0.015$
$\mathcal{P}(X < 110)$	0.71697	0.70071	0.67270	0.84094	0.82414	0.79374
$\mathcal{P}(X < 90)$	0.60125	0.58283	0.55175	0.78053	0.76019	0.72411
$\mathcal{P}(X < 80)$	0.52804	0.50914	0.47761	0.73966	0.71743	0.67847
$\mathcal{P}(X < 60)$	0.35030	0.33289	0.30467	0.62577	0.60009	0.55637
$\sigma = 6$						
	$\alpha = 0.007$	$\alpha = 0.01$	$\alpha = 0.015$	$\alpha = 0.007$	$\alpha = 0.01$	$\alpha = 0.015$
$\mathcal{P}(X < 110)$	0.63337	0.61174	0.57495	0.75993	0.73420	0.68843
$\mathcal{P}(X < 90)$	0.48406	0.46141	0.42397	0.66889	0.63921	0.58792
$\mathcal{P}(X < 80)$	0.39611	0.37435	0.33899	0.60981	0.57861	0.52558
$\mathcal{P}(X < 60)$	0.20918	0.19319	0.16826	0.45734	0.42558	0.37380
$\sigma = 7$						
	$\alpha = 0.007$	$\alpha = 0.01$	$\alpha = 0.015$	$\alpha = 0.007$	$\alpha = 0.01$	$\alpha = 0.015$
$\mathcal{P}(X < 110)$	0.56076	0.53446	0.49035	0.68357	0.64943	0.58989
$\mathcal{P}(X < 90)$	0.38541	0.36028	0.31975	0.56498	0.52778	0.46540
$\mathcal{P}(X < 80)$	0.29046	0.26814	0.23296	0.49172	0.45435	0.39312
$\mathcal{P}(X < 60)$	0.11725	0.10469	0.08598	0.31880	0.28605	0.23530

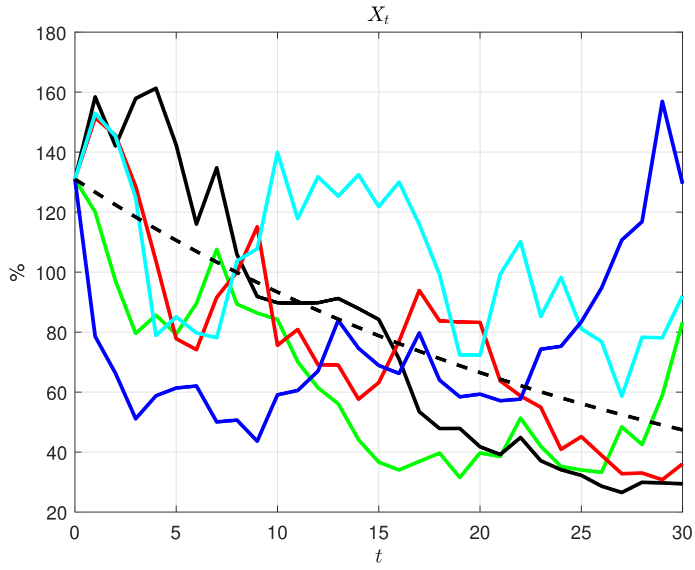
Note: the table reports the probability that debt ratio is below a certain threshold level for different combinations of uncertainty, σ, and fundamentals, α, given an initial debt ratio $x = 130\%$.

Table 4: Safe Debt as a Minimal Condition for Fiscal Solvency under the Optimal Solution

	$\alpha = 0.007$	$\alpha = 0.01$	$\alpha = 0.015$
$\sigma = 5$	105.3256 (99.7368,170)	106.4170 (101.2441, 170)	108.3730 (103.8071, 170)
$\sigma = 6$	113.5887 (109.3114,170)	114.8776 (110.9635, 170)	117.1689 (113.7725, 170)
$\sigma = 7$	119.0586 (115.5234,170)	120.4676 (117.2693, 170)	122.9672 (120.2380, 170)

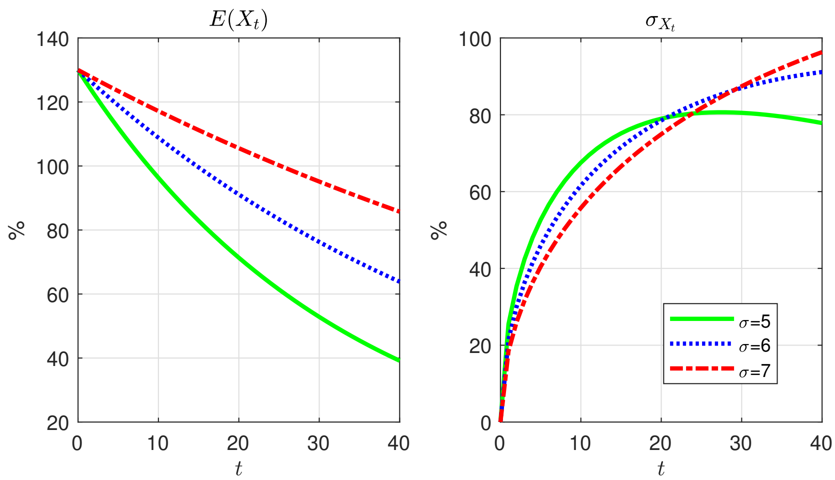
Note: the table reports the initial level of debt, x, and the related boundaries, $\underline{x}$ and $\overline{x}$, such that, over a 5-year time horizon, the probability $\mu^x(\underline{x})$ is equal to 0.95.

Figure 1: Debt Ratio - Sample Paths



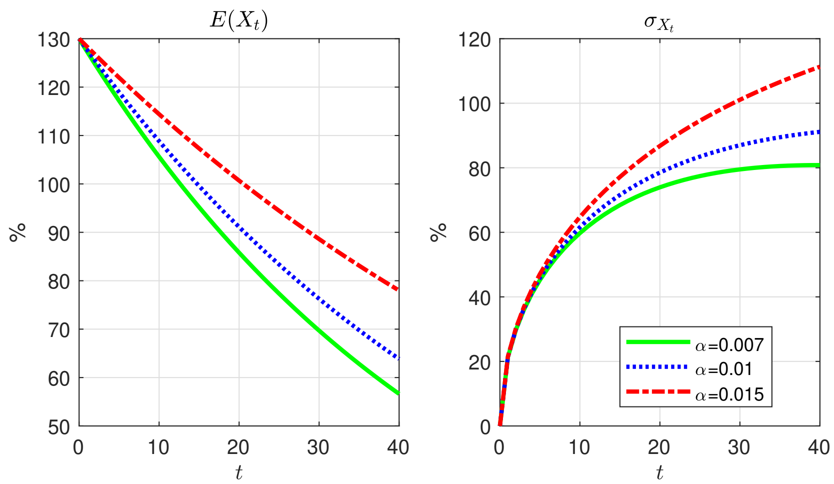
Note: the figure plots five sample paths for the debt ratio X_t, given an initial debt ratio $x = 131.3\%$, $\alpha = 0.0066$ and $\sigma = 6.086$. The dashed line is the deterministic time path of the debt ratio under the same fiscal effort.

Figure 2: Debt Ratio and Uncertainty - Theoretical Moments



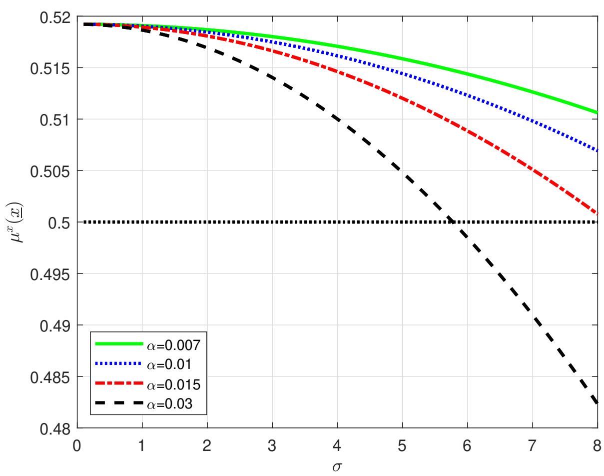
Note: the figure plots the mean and the standard deviation of the debt ratio X_t for different values of uncertainty, σ, given an initial debt ratio $x = 130\%$ and fundamentals $\alpha = 0.01$.

Figure 3: Debt Ratio and Fundamentals - Theoretical Moments



Note: the figure plots the mean and the standard deviation of the debt ratio X_t for different values of fundamentals, α, given an initial debt ratio $x = 130\%$ and uncertainty $\sigma = 6$.

Figure 4: Fiscal Resilience, Uncertainty, and Fundamentals



Note: the figure plots $\mu^x(\underline{x})$ for different values of uncertainty, σ, and fundamentals, α, given an initial debt-ratio gap $x = 130\%$ and interval boundaries $\overline{x} = 135\%, \underline{x} = 125\%$.

Appendix A

In this appendix, we shortly discuss the concept of *filtration* and *Markov control* to be used in the optimal control problem solved in Appendix B. The reader may refer to Øksendal (2003), Section 11.1 for more details.

Definition. Given the *measurable space* $(\Omega, \mathcal{F})$, the (increasing) family $\{\mathcal{M}_t\}_{t\geq 0}$ of $\sigma - algebras$ of Ω such that

$$\mathcal{M}_{t_1} \subset \mathcal{M}_{t_2} \subset \mathcal{F}\,, \qquad \forall\; 0 \leq t_1 < t_2\,,$$

is called a *filtration* on $(\Omega, \mathcal{F})$. □

Since for every t we have a random control variable which the random variable X_t depends upon, we consider a complete probability space Ω with filtration $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, where the filtration $\mathcal{F}_t$ is the one generated by the standard one-dimensional Brownian motion W_t and is augmented by $\mathbb{P}$-null sets, that is

$$\mathcal{F}_t = \sigma\Big(W_s,\;\; 0 \leq s \leq t\Big) \cup \Big\{A \in \mathcal{F} \,|\, \mathbb{P}(A) = 0\Big\}, \qquad \forall\; t \geq 0. \tag{A.1}$$

Moreover, we assume that X_0 is an integrable random variable with law π_0 and measurable with respect to $\mathcal{F}_0$ representing the initial value of the current debt-to-output ratio. For the sake of simplicity, we assume that at the time 0 the debt-ratio is deterministic, so that $X_0 = x$ is a constant with $x > 0$.

The control variable $\rho_t = \rho(t, \omega)$ is taken from a given family $\mathcal{A}$ of admissible controls:

$$\begin{aligned}\mathcal{A} := \Big\{&\rho_t = \rho(t, X_t(\omega)) \text{ for some Borel-measurable} \\ &\text{bounded functions } \rho : [0,T] \times [0,\bar{x}] \longrightarrow [0,\bar{\rho}]\Big\}\end{aligned} \tag{A.2}$$

and is adapted to the filtration (A.1).

At time t the value of the function $\rho(t, X_t)$ only depends on the state of the system at that time, thus it does not depend on the probability space ω explicitly, but only through the process X_t. Such a ρ is called *Markov control* and the corresponding process X_t becomes an Itô diffusion, in particular a Markov process.

Appendix B

To apply the HJB method we have to transform the mean value J_T, given in (9), into the mean value of an integral by relying on the Dynkin's formula. According to the Dynkin's formula the mean value J_T, given in (9), reads, for $X^{s,x}_{t-s} \equiv X_{t-s}$

$$J(s,x;\rho) = x^2 + \mathbf{E}\left[\int_s^T \{[\alpha X_{t-s} - \rho_{t-s} X_{t-s}](2X_{t-s}) + \right. \\ \left. + \sigma^2 \rho^2_{t-s} X^2_{t-s}\Big\}\, dt\right], \tag{B.1}$$

where T is the stopping time introduced in (10). By virtue of the invariance of the problem under time translation (that is, the homogeneous problem over time), we can rewrite $J(s,x;\rho)$, given in (B.1), in the form:

$$J(s,x;\rho) = x^2 + \mathbf{E}\left[\int_0^{T-s} \Big\{[\alpha X_t - \rho_t X_t](2X_t) + \sigma^2 \rho_t^2 X_t^2\Big\}\, dt\right]. \tag{B.2}$$

where we denote the process $(X_t^{0,x})_{t \le T}$ with X_t. Then the optimization problem can be equivalently written in the form:

$$\begin{cases} \phi(s,x) := x^2 + \inf\limits_{\rho_t \in \mathcal{A}} \mathbf{E}\left[\int_0^{T-s} \{[\alpha X_t - \rho_t X_t](2X_t) + \sigma^2 \rho_t^2 X_t^2\}\, dt\right], \\ \text{sub} \qquad dX_t = [\alpha X_t - \rho_t X_t]dt - \sigma \rho_t\, X_t\, dW_t, \qquad \text{with} \quad X_0 = x, \end{cases} \tag{B.3}$$

where the initial value $X_0 = x$ is fixed in order the equation for X_t to have an unique solution.

Note that in choosing the optimal value for ρ_t, the government will take into account two relevant elements: first, the corrective power of the primary surplus, second, the relative volatility that the fiscal action itself may transmit to the system. As a result, the optimal solution of ρ_t is expected to strike a balance between these two opposite forces, as any effort in reducing debt is conducive to uncertain feedback effects for the system.

By applying the HJB equation to the second term on the right hand side of the optimization problem in (B.3), we obtain the following variational equation for $w \equiv \rho_t$:

$$\inf_w \left\{[2(\alpha - w) + \sigma^2 w^2] X_t^2 + \frac{\partial \phi}{\partial t} + (\alpha - w) X_t \frac{\partial \phi}{\partial x} + \frac{1}{2}\left(\sigma^2 w^2 X_t^2\right) \frac{\partial^2 \phi}{\partial x^2}\right\} = 0. \tag{B.4}$$

To find an optimal control we now derive equation (B.4) with respect to w and for $X_t = x$ we obtain the following equation after simplification of a factor x:

$$-2x + 2\sigma^2 w x - \frac{\partial \phi}{\partial x} + \sigma^2 w x \frac{\partial^2 \phi}{\partial x^2} = 0, \tag{B.5}$$

from which it immediately follows:

$$w \equiv \rho_t = \frac{2x + \dfrac{\partial \phi}{\partial x}}{\sigma^2 x \left(2 + \dfrac{\partial^2 \phi}{\partial x^2}\right)}. \tag{B.6}$$

To find a solution for equation (B.6) we try with a guess function with separated variables of the following type:

$$\phi(s,x) = c\,x^2\,g(s), \tag{B.7}$$

where $g(0)$ is a constant, that is $g(0) = K$, because the value function $\phi(s,x)$ in the optimization problem (B.3), as we shall see, is a multiple of x^2, that is it yields $\phi(0,x) = \tilde{K}x^2$. Note that HJB equation is a parabolic equation, and then the condition $\phi(0,x) = \tilde{K}x^2$ is the initial value, but we can not impose any boundary conditions because x is already fixed by the initial condition of the evolution equation in (B.3).

By substituting (B.7) into (B.6), where $\partial\phi/\partial x = 2cxg(s)$ and $\partial^2\phi/\partial x^2 = 2cg(s)$, we obtain:

$$\hat{\rho}_t = \hat{\rho}(t, X_t(\omega)) = \hat{\rho} = \frac{1}{\sigma^2}, \tag{B.8}$$

that is equation 13 of the main text and shows that the optimal Markov control $\hat{\rho}(t,X)$ is constant.

Note that the constant control (B.8) is admissible as it belongs to the set $\mathcal{A}$ as in (A.2). Indeed a constant function is a Borel-measurable function ρ belonging to the set in (A.2). See Appendix A. Sufficient conditions ensuring that the optimization problem (B.3) satisfies the requirements for the optimality of the optimal Markov control solution are in Appendix C, where we show that the application which associates $cx^2g(s)$ with every pair (s,x) satisfies all the conditions of the *Verification Theorem.*

Appendix C

In this appendix, we give sufficient conditions to conclude that (13) is the optimal Markov control process and how the corresponding value function is. The proof relies essentially on Itô's lemma as follows.

First we give and explicit expression of the value function $\phi(s,x)$. To find it we have to substitute the expressions (B.7) and (13) into equation (B.4) in order to obtain the following separated equation for the temporal function $g(s)$:

$$g'(s) + \left(2\alpha - \frac{1}{\sigma^2}\right) g(s) = \frac{1}{c}\left(\frac{1}{\sigma^2} - 2\alpha\right). \tag{C.1}$$

If we substitute the solution (15) into the integral in (B.3), the value function, for $s = 0$, becomes:

$$\phi(0,x) = x^2 + (2\alpha - 1/\sigma^2)x^2 \ \mathbf{E}\left[\int_0^T e^{(2\alpha-3/\sigma^2)t - 2W_t/\sigma} dt\right] = \tilde{K}x^2,$$

because the mean value of a random variable is obviously a constant.

If we now consider the initial condition $g(0) = K \equiv \tilde{K}/c$, the solution of the temporal ordinary differential equation (C.1) reads:

$$g(s) = \left(K + \frac{1}{c}\right) e^{\left(1/\sigma^2 - 2\alpha\right)s} - \frac{1}{c}. \tag{C.2}$$

We then have an explicit expression of the value function $\phi(s,x)$ given by $\phi(s,x) = \left[(Kc+1)\, e^{\left(1/\sigma^2-2\alpha\right)s} - 1\right] x^2$.

Verification Theorem

Let

$$\phi(s,x) := \inf_{\rho_t \in \mathcal{A}} \mathbf{E} \int_0^{T-s} f(t, X_t, \rho_t)\, dt,$$

with

$$dX_s = b(s, X_s, \rho(s, X_s))ds + \sigma(s, X_s, \rho(s, X_s))dW_s \quad \text{and} \quad X_0 = x,$$

be an optimisation problem. Let V be a $C^{1,2}([0,T)\times\mathbb{R}) \cap C([0,T)\times\mathbb{R})$ function and let us assume that f and V have quadratic growth, i.e. there is a constant C such that

$$|f(t,x,\rho)| + |V(t,x)| \leqslant C(|x|^2 + 1), \tag{C.3}$$

for all $(t,x,\rho) \in [0,T) \times \mathbb{R} \times \mathcal{A}$. (i) Suppose that

$$\frac{\partial V(t,x)}{\partial t} + f(t,x,\rho) + b(t,x,\rho)\,\frac{\partial V(t,x)}{\partial x} + \frac{\sigma^2(t,x,\rho)}{2}\,\frac{\partial^2 V(t,x)}{\partial x^2} \geqslant 0 \tag{C.4}$$

on $[0,T)\times\mathbb{R}$. Then $V \leqslant \phi$ on $[0,T]\times\mathbb{R}$. (ii) Assume further that there exists a minimizer $\hat{\rho}(t,x)$ of the function

$$\rho \to \mathcal{L}^\rho V(t,x) + f(t,x,\rho),$$

such that

$$
\begin{aligned}
0 &= \frac{\partial V(t,x)}{\partial t} + \inf_{\rho\in\mathcal{A}} \left\{ f(t,x,\rho) + b(t,x,\rho)\frac{\partial V(t,x)}{\partial x} + \frac{\sigma^2(t,x,\rho)}{2}\frac{\partial^2 V(t,x)}{\partial x^2} \right\} = \\
&= \frac{\partial V(t,x)}{\partial t} + \mathcal{L}^{\hat{\rho}(t,x)} V(t,x) + f(t,x,\hat{\rho}),
\end{aligned}
\tag{C.5}
$$

where $\mathcal{L}^{\rho} V(t,x)$ is defined as

$$
\mathcal{L}^{\rho} V(t,x) := b(t,x,\rho)\frac{\partial V(t,x)}{\partial x} + \frac{\sigma^2(t,x,\rho)}{2}\frac{\partial^2 V(t,x)}{\partial x^2} \,. \tag{C.6}
$$

Then the stochastic differential equation

$$
dX_s = b(s, X_s, \hat{\rho}(s, X_s))ds + \sigma(s, X_s, \hat{\rho}(s, X_s))dW_s \tag{C.7}
$$

defines a unique solution X for each given initial date $X_0 = x$ and the process $\hat{\rho} := \hat{\rho}(s, X_s)$ is a well-defined control process in $\mathcal{A}$. Then ϕ is the value function and $\hat{\rho}$ is the optimal Markov control process. In our case from equation (B.3) we have

$$
f(t,x,\rho) = \left[\sigma^2(1+\alpha)^2\rho^2 + 2\alpha - 2\rho(1+\alpha)\right] x^2, \tag{C.8}
$$

which has quadratic growth and then it follows that there exists a positive constant such that

$$
C_1 \geqslant \sup_{\rho} \left\{ |\sigma^2(1+\alpha)^2\rho^2 + 2\alpha - 2\rho(1+\alpha)| \right\} \tag{C.9}
$$

and $|f(t,x,\rho)| \leqslant C_1 x^2$. Since the term in square brackets in ((C.8)) is bounded, it follows that there exists a positive constant C_2 such that $|f(t,x,\rho)| \leqslant C_1 x^2$. Then, the condition (C.3) is satisfied with a positive constant $C > C_1 + C_2 - 1$. Further, the condition (C.4) is verified, too, because the expression

$$
\frac{\partial V(t,x)}{\partial t} + f(t,x,\rho) + b(t,x,\rho)\frac{\partial V(t,x)}{\partial x} + \frac{\sigma^2(t,x,\rho)}{2}\frac{\partial^2 V(t,x)}{\partial x^2} \tag{C.10}
$$

is convex with respect to ρ and thus positive or null for each $\rho \neq \hat{\rho}$. Then, according to the Verification Theorem, the value function is thus $\phi(s,x)$, and the optimal policy is the constant process $\hat{\rho}_t$.

Appendix D

In this appendix, we derive equation (22) by means of Dynkin's formula. Specifically, let

$$dX_t = r(X_t)dt + \sigma(X_t)dB_t \tag{D.1}$$

be a 1-dimensional Itô diffusion with characteristic operator $\mathcal{A}$ and $f \in C^2(\mathbb{R})$ be a solution of the ordinary differential equation:

$$\mathcal{A}f(x) = r(x)f'(x) + \frac{\sigma^2(x)}{2} f''(x) = 0, \quad x \in \mathbb{R}. \tag{D.2}$$

Let $(a, b) \subset \mathbb{R}$, with $b > a > 0$, be an open interval such that $x \in (a, b)$ and put

$$\tau \equiv \tau(a, b) = \inf\{t > 0 \text{ such that } X_t \notin (a, b)\},$$

and assume that $\tau < \infty$ a.s. with respect to the probability law of X_t. If we define

$$p \equiv P^x[X_\tau = b],$$

it follows

$$p = \frac{f(x) - f(a)}{f(b) - f(a)}\,. \tag{D.3}$$

Proof. If we consider the function $f_0 \in C_0^2(\mathbb{R})$ such that $f_0(x) \equiv f(x)$ on (a, b) and $\mathcal{A}f_0(x) = \mathcal{A}f(x) = 0$, by means of Dynkin formula we can write:

$$E^x[f(X_\tau)] = f(x) + E^x\left[\int_0^\tau \mathcal{A}f(X_s)\,ds\right] = f_0(x). \tag{D.4}$$

Since $f_0(x) \in C_0^2(\mathbb{R})$ and $X_{\tau(a,b)} \notin (a, b)$, it follows that the random variable $X_{\tau(a,b)}$ can assume the two values a and b, only. Then the mean value $E^x[f(X_\tau)]$ of $f(X_\tau)$ is given by the sum of the two products of the values $f_0(a)$ and $f_0(b)$ multiplied by the corresponding probabilities, $1 - p$ and p respectively, that is

$$f_0(x) \equiv E^x[f(X_\tau)] = f_0(a)(1 - p) + f_0(b)p. \tag{D.5}$$

From the equality between the first and the third term we obtain the final relation

$$p(b) = \frac{f(x) - f(a)}{f(b) - f(a)}, \tag{D.6}$$

and thus:

$$p(a) = \frac{f(b) - f(x)}{f(b) - f(a)}, \tag{D.7}$$

because the equality on the boundary of the interval $f_0(a) = f(a)$ and $f_0(b) = f(b)$ hold. In the text we assume that $a = \underline{x}$ and $b = \overline{x}$ from which it follows equation (22).

Appendix E

In order to give an explicit expression to (22) we have first to solve equation (20). To this aim we transform it into a first-order differential equation through the change of the variable $f'(x) = g(x)$, so that our equation now reads:

$$\frac{x^2}{2\sigma^2}\frac{dg(x)}{dx} + \left(\alpha - \frac{1}{\sigma^2}\right) x\, g(x) = 0. \tag{E.1}$$

By separating $x, g(x)$ one obtains

$$\frac{dg}{g} = \left(2 - 2\sigma^2\alpha\right)\frac{dx}{x}, \tag{E.2}$$

whose solution is

$$g(x) = (f')^{2-2\sigma^2\alpha}\,. \tag{E.3}$$

By integration, we finally obtain the function $f(x)$:

$$f(x) = \int g(x)\, dx = C\left(\frac{x^{3-2\sigma^2\alpha}}{3 - 2\sigma^2\alpha}\right) + K. \tag{E.4}$$

We now have an explicit expression of the harmonic measure that debt hits the extremes of D as follows:

$$\mu^x(\underline{x}) = \frac{\overline{x}^{3-2\sigma^2\alpha} - x^{3-2\sigma^2\alpha}}{\overline{x}^{3-2\sigma^2\alpha} - \underline{x}^{3-2\sigma^2\alpha}}, \tag{E.5}$$

$$\mu^x(\overline{x}) = 1 - \mu^x(\underline{x}). \tag{E.6}$$