Optimal Correction of the Public Debt and Measures of Fiscal Soundness*

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April 2021

Abstract

This paper derives the optimal response of the primary budget balance to changes in the public debt a share of gross domestic product (GDP) in a stochastic model of debt. Under the optimal solution the surplus reactivity to the debt-GDP ratio is independent of the debt ratio itself, but its size depends on the degree of uncertainty surrounding the impact of fiscal policies. We characterize the properties of the optimal control policy proposing different metrics that may be used to assess fiscal soundness and as early warning indicators of fiscal imbalances.

Keywords: Debt-GDP Ratio, Optimal Control, Fiscal Consolidation, Resilience.
JEL Classification Codes: H62, H63, E63.

*The views expressed are purely those of the authors and may not in any circumstances be regarded as stating an official position of the European Commission. We are very grateful to Ludovic Cales, Roberta Cardani, Olga Croitorov, Maria Rita Ebano, Giorgio Ferrari, Lorenzo Frattarolo, Massimo Giovannini, Alessandra Pelloni, Filippo Pericoli, Marco Ratto, Alessandro Rossi and to seminar participants at the JRC and Sapienza University for helpful comments and discussions. An earlier version of this paper circulated under the title "Optimal Correction of the Public Debt and Fiscal Resilience Measures".

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Declarations

Declarations of interest: none.

Funding

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Data Availability

No data were used. Matlab codes available upon request.

1 Introduction

In the last decade and a half since the Great Recession and the global financial crisis, dramatic fiscal developments have brought renewed attention on the issue of public debt sustainability. The economic crisis triggered by the SARS-CoV-2 outbreak has brought about a sharp increase of public debt in Europe. Since the onset of the health emergency public debt-GDP ratios have been soaring to levels never seen in any year during and after the 2008-2009 crisis. The extreme economic slowdown is requiring an unprecedented fiscal stimulus, although at the end of this very unusual period sustainability issues may emerge heavily, especially when financial markets will start again to price the risk of default. Nonetheless, the expensive legacy of the pandemic will push governments to find the right path between fiscal support to ward off a second economic downturn, and proper restraints to safeguard fiscal solvency. This issue has turned out to be of particular concern for heavily indebted countries of the euro area, where the risk of a debt overhang has already kicked in during the last decade. This is why the issues of public debt control and of sustainability analysis are gaining momentum once again.

In this respect, the exceptionally large uncertainty surrounding economic developments, including the effects of fiscal actions and the evolution of public debt, calls for the development of new tools of analysis that allow us to study the issue of the debt-GDP ratio control in a stochastic environment, and that provide measures of fiscal solvency and stability that may come in handy as early warning indicators in the surveillance of fiscal imbalances and as indicators of the soundness of consolidation packages. To this end this paper proposes a continuous time stochastic model of debt where a budget rule automatically triggers a correction mechanism of the primary balance to the debt ratio. The problem is that of a government wishing to cut down the current level of debt ratio, while uncertainty comes through shocks that may frustrate or magnify the effects of the fiscal rule itself, given the existence of interdependencies between the fiscal stance and other determinants of debt accumulation, such as interest rates and economic growth.

By relying on optimal control theory and applying the Hamilton-Jacobi-Bellman equation, we show that under the optimal Markov control the relationship between the primary balance and the debt ratio is linear.[1] The optimal reactivity of the primary balance to the debt ratio is decreasing in the degree of uncertainty surrounding the effects of fiscal policies. Thus the optimal correction rule prescribes a more vigourous response when the impact of fiscal consolidation is less uncertain.

Given the optimal control we characterize the properties of the solution and propose different metrics that can be used to assess the soundness of a consolidation plan and as early warning indicators of fiscal imbalances in the presence of uncertainty. This is a useful exercise to single out the role of uncertainty in setting the optimal control and identify the major factors that may undermine

[1]For a study in which the objective of the government is, instead, that of keeping the level of output closer to a reference value in the attempt of stabilizing the economy over the business cycle, see Correani et al. (2014), who use optimal control theory and apply the Hamilton-Jacobi-Bellman equation in a stochastic IS-LM model.

the achievement of a fiscal target. One measure simply refers to the time needed to reach an expected debt ratio. Clearly, the longer the time interval necessary to reduce the debt ratio by a given amount, the lower the soundness of the fiscal policy put in place. Nonetheless, the time necessary to reach a targeted amount will be affected by GDP growth, the rate of return on debt, and the uncertainty surrounding the effects of fiscal policy. A second measure is instead based on the probability to reach a fiscal consolidation objective in a given time interval. The third measure we propose is the probability that the debt ratio, in case of its exit from a given interval as a result of shocks, will not depart "too much" from it. The idea is to gauge the fiscal soundness in the case that adverse shocks are able to push the debt ratio out from a given interval. In this context a high probability would reflect good resilience to outside shocks, and so would be a sign of fiscal soundness. This measure relies on the concept of harmonic measure, describing the probability distribution of the debt ratio as it hits the boundary of a given open interval.[2] As far as we know we are the first to employ an approach based on a harmonic measure to construct a measure of fiscal resilience.

We argue that the all the three proposed measures may be fruitfully used as indicators of the goodness of a fiscal package and as pre-alert indicators of fiscal imbalances. In particular, the first two measures are more appropriate to evaluate the health of public finances in a medium- and long-run perspective, while the third measure is more appropriate to assess the government's ability to service all the upcoming obligations in a short-run perspective.

This paper is related to the vast literature on debt sustainability assessment and on measures of fiscal soundness. In the last decades this literature has evolved to account for the fact that debt sustainability analysis requires awareness of the uncertainty surrounding the evolution of public debt.[3] This strand of literature, mostly developed at institutional level and within international organizations, explicitly accounts for the fact that fiscal solvency and debt behaviour depend on the future dynamics of economic fundamentals that are not known for sure and that may be extremely volatile (e.g. Berti 2013, Rozenov 2017 and Cherif and Hasanov 2018), highlights the importance of designing fiscal rules that are truly operational (see Eyraud et al. 2018), and proposes methods to quantify the fiscal stress (e.g. Balducci et al. 2011 and Pamies Sumner and Berti 2017) and the fiscal space (e.g. Ghosh et al. 2013).

From a methodological point of view, the closest predecessors of our paper are those dealing with stochastic control problems of the debt-GDP ratio. In particular, Ferrari (2018) explores the case of a government whose objective is that of reducing the debt ratio through the minimization of two opposing costs, namely the expected opportunity cost of having debt on the one hand, and the expected cost from the reduction policy on the other hand. In more detail, Ferrari (2018)

[2]The theory of harmonic measure has been extensively used in several applications, such as the corona problem and in mapping problems. It has particularly interesting applications in probability theory, especially in relation to Brownian motions. See Garnett and Marshall (2005) for a survey of the theory and applications concerning this measure.

[3]For recent comprehensive overviews of the tools commonly used to measure public debt sustainability and anticipate fiscal vulnerabilities, see Corsetti (2018) and Debrun et al. (2019).

shows that the solution of the control problem is related to that of an auxiliary optimal stopping problem. Put it differently, dealing with the optimal stopping problem is equivalent to work out the solution of the corresponding control problem. In conclusion, the optimal policy is found to be that of keeping the debt ratio under an inflation-dependent ceiling. Ferrari and Rodosthenous (2018) introduce the problem of a government managing the debt ratio in a stochastic continuous time model where uncertainty comes through a macroeconomic risk process affecting the interest rate bearing on public debt. The exogenous risk process is modelled as N-state continuous-time Markov chain, while the government faces a trade-off between the potential benefits from high public investments and the costs deriving from having an excessive debt ratio and austerity policies. At the optimum the government would keep the debt ratio in an interval whose boundaries depend on the possible states of the Markov process. Callegaro et al. (2020) study the problem of a government aiming at reducing the debt ratio under partial information where the underlying macroeconomic conditions are not directly observed. Cadenillas and Huamán-Aguilar (2016) develop a stochastic debt control model to find the optimal ceiling for the government debt. As in Ferrari (2018) the government objective is that of minimising the trade-off between the opportunity cost of having debt and the cost from arising from its reduction. Cadenillas and Huamán-Aguilar (2016) obtain a closed-form solution for the optimal government debt ceiling and find that the fiscal policy will be active if the debt ratio is greater than the optimal debt ceiling, while a passive fiscal policy will be desirable if debt is lower than the ceiling.[4] In a subsequent paper Cadenillas and Huamán-Aguilar (2018) study the optimal debt ceiling accounting for the fact that the ability of the government to reduce its debt ratio is bounded.

The remainder of the paper is structured as follows. Section 2 lays out the model. Section 3 introduces and solves the optimisation problem of the government. Section 4 presents different measures of fiscal soundness under the optimal control policy. Section 5 presents concluding remarks.

2 The Model Setup

A simple starting point for the formal discussion of public finances is the flow budget constraint of the government which dictates that the next period debt is given by the current period debt times a gross interest factor minus the primary balance (government revenues minus expenditures excluding interest payments). This relationship can be easily written in terms of GDP share as follows:

$$X_{t+1} = \frac{1 + r_t}{1 + g_t} X_t - S_t, \tag{1}$$

[4]In a previous contribution Huamán-Aguilar and Cadenillas (2015) propose a stochastic model for government debt control under the assumption that debt may also be issued in foreign currency. They show that for high debt aversion and exchange rate uncertainty, it is optimal to reduce the share of the debt burden denominated in foreign currency in favour of domestic currency.

where X is the the stock of public debt as a proportion of GDP, S is the primary balance as a share of GDP, r denotes the interest rate on government debt, and g is the growth rate of GDP.

By making use of the approximation $\frac{1+r}{1+g} \approx 1 + r - g$, equation (1) can be equivalently expressed as follows:

$$X_{t+1} = X_t(1 + \alpha_t) - S_t. \tag{2}$$

where $\alpha_t \equiv r_t - g_t$.

We assume that the behaviour of the primary balance is described by a debt-based reaction rule.[5] In particular, we focus on a reaction rule to the debt ratio of the form:

$$S_t = \rho_t X_t, \tag{3}$$

where ρ_t is the government control variable measuring the strength of the primary surplus response to the debt ratio X_t. The above rule simply prescribes that the primary surplus is a function of the debt ratio. A positive response of the primary surplus implies that a government is taking corrective actions that counteract the changes in debt, consistently with empirical evidence.[6]

Given (3), equation (2) can be re-written as:

$$X_{t+1} - X_t = (\alpha_t - \rho_t)\, X_t, \tag{4}$$

Given the uncertain feedback effects from the fiscal adjustments to economic variables, α_t is assumed to have two components, a deterministic component capturing long-run fundamentals, and a random factor that depends on the degree of the fiscal effort ρ_t. In particular, we assume that

$$\alpha_t = \alpha - \sigma \rho_t \epsilon_t, \tag{5}$$

where the term α reflects the long term interest-growth differential, σ represents the diffusion coefficient meant to transmit uncertainty to the response action of policy makers and ϵ_t is the white noise in discrete time.

The second term in (5) captures the uncertainty that may surround the final outcome of any fiscal intervention and is related to the macroeconomic effects of fiscal policy on the interest-growth differential. Clearly, this stochastic component affects the effective size of the primary balance ratio. This uncertainty may originate from several factors and macroeconomic interdependency mechanisms.[7] An ambitious fiscal consolidation plan may deteriorate economic conditions to such an extent that tax revenues decline and social spending increases, partially

[5] This rule is basically consistent with the requirements laid down in the Fiscal Stability Treaty. According to this agreement, member states are expected to narrow the difference between the actual government debt-GDP ratio and 60% of GDP at an average rate of one-twentieth per year. However, in the long term member states are asked to adjust their fiscal policies, namely their primary balance, in order to meet the ultimate target of 60% of GDP.

[6] See e.g. Bohn (1998).

[7] See also Balibek and Köksalan (2010) for a model of debt management taking into account the uncertainty concerning the future state of the economy.

frustrating the initial correction. This self-defeating mechanism of the corrective measure may thus lead to a negative shock.[8] Similarly, a strong corrective fiscal intervention may undermine growth prospects, pushing private investors to cut down their investment plans, leading to a knock-on effect to the level of economic activity and thus to the debt-GDP ratio. Different beliefs about the type of fiscal consolidation may give rise to waves of optimism that may improve the performance of the consolidation itself or, alternatively, to waves pessimism that may magnify the contractionary effects the ongoing specific fiscal plan.[9] However, debt reduction may be also conducive to positive shocks. Indeed, a surplus correction may increase the confidence of private investors, boosting market confidence and lowering risk premia in countries with high debt, so that we may observe a positive effect on the surplus ratio. Further, possible non-Keynesian effects of fiscal policy may give rise to beneficial effects on the budget balance by magnifying the effects of a fiscal intervention.

A credible fiscal consolidation plan can signal future reductions of distortionary taxes and, therefore, an increase in permanent income, producing an increase in private consumption. Private investments may also respond positively, via the interest rate channel or an expected lower tax burden in the future.[10] Non-Keynesian effects of fiscal policy could then account for positive shocks. According to the empirical evidence discussed in Alesina et al. (2018), fiscal corrective measures based upon spending adjustments are much less costly in terms of output losses than those based upon tax adjustments. In general, the stochastic component in (5) is meant to capture the dilemma faced by fiscal authorities that must strike a balance between the need of a strong action and the uncertainty that the action itself may magnify.

To analyze how the debt accumulation evolves in the presence of a stochastic term, we substitute the fiscal rule (5) into equation (4) and obtain the following expression in discrete time

$$X_{t+1} - X_t = (\alpha - \rho_t) X_t dt - \sigma \rho_t X_t \epsilon_t \,, \tag{6}$$

which in continuous time becomes:

$$dX_t = (\alpha - \rho_t) X_t dt - \sigma \rho_t X_t dW_t \,, \tag{7}$$

where W_t is an one-dimensional Brownian motion with zero mean and density function given by a Gaussian exponential law of the type:

[8] See e.g. DeLong et al. (2012). According to empirical evidence, fiscal multipliers are large during recessions and small when the economy operates close to potential. See Auerbach and Gorodnichenko (2012) and Corsetti et al. (2013). On the positive effects of fiscal expansion see e.g. Blanchard and Perotti (2002).

[9] The effects of fiscal actions also depend on the underlying monetary-fiscal policy regime, on expectations about future regimes and on the credibility of an announced fiscal plan. All these factors are not directly controlled by policymakers. In this respect, for a comprehensive discussion on how "darned hard" fiscal analysis is, see Leeper (2015).

[10] See e.g. Giavazzi and Pagano (1990) and Alesina and Ardagna (2013), and the discussion in Padoan et al. (2013).

$$W_t \sim \frac{e^{-\frac{y^2}{2t}}}{\sqrt{2\pi t}} \tag{8}$$

and we have identified the term $\epsilon_t \, dt$ in (6) with dW_t in (7).

The variable described by equation (7) is an Itô process with a unique solution, since it satisfies the two conditions for the existence and uniqueness of the solution.

For more details about these conditions and the mathematical features underlying equation (7), see Øksendal (2003) and Appendix A.

3 The Optimisation Problem

In this section we consider the problem of a government aiming at reducing the current level of the debt ratio. Moreover, the fiscal authority is assumed to have always access to the available policy tool ρ, that is the strength of the primary balance response to the debt ratio. The government is assumed to be increasingly worse-off the larger the debt ratio. The idea is that the government faces an instantaneous loss related the rising of the public debt. Notably, a large public debt may crowd out private investment undermining growth prospects.[11] In addition one of the potential effects associated with an excessive public debt is that of an increase in the perceived risk that a country may default on its debt. This change in market sentiments may push an economy towards a bad equilibrium through self-fulfilling upward effects on yields and debt may become unsustainable. Moreover, since the unpleasant arithmetic of Sargent and Wallace (1981) it has been well known that it is impossible for a monetary authority to sustain low inflation in the presence of excessive public debt and profligate fiscal policy. Finally, the implementation of restrictive fiscal policies in response to an increase in the debt ratio may hinder growth, especially during a recession (see DeLong et al. 2012).

This assumption translates in a quadratic expected loss function J_T of the type:

$$J_T = \mathbf{E}\left[(X_T)^2\right] \equiv \int_\Omega (X_T)^2 \, d\mathbb{P}, \tag{9}$$

where X_T is the stochastic level of debt at time T and $\mathbf{E}$ denotes the expectation value with respect to the probability law of X, that is with respect to the probability measure $\mathbb{P}$.

The time T is the *exit time* of the process X_t from its interval $(\underline{x}, \overline{x})$, that is it yields

$$T = \inf_t \{t > 0 \text{ such that } X_t \leqslant \underline{x}\}, \tag{10}$$

with $\mathbf{E}[T] < \infty$.

[11]There is a quite vast empirical literature which shows that there is a negative correlation between public debt and economic growth (see e.g. Reinhart and Rogoff 2010, Woo and Kumar 2015). Yet, the casual interpretation of the correlation is an open issue since there might be cases in which causation goes from low growth to high debt, rather than the other way round.

As usual in the dynamic programming literature, we now let the controlled diffusion X start at time s from level $x > 0$, that is

$$\begin{cases} dX_t^{s,x} = \left(\alpha - \rho_{t-s}\right) X_{t-s}\, dt - \sigma \rho_{t-s} X_{t-s} dW_{t-s}, \\ \text{sub} \qquad X_s^{s,x} = x. \end{cases} \tag{11}$$

The optimization problem now reads:

$$\phi(s,x) = \inf_{\rho \in \mathcal{A}} \mathbf{E}\left[(X_T^{s,x})^2\right], \tag{12}$$

where $\phi(s,x)$ denotes the value function. To solve the system, we use the Hamilton-Jacobi-Bellman (HJB) equation.[12]

Nevertheless, to apply the HJB method we have to preliminarily transform the mean value J_T, given in (9), into the mean value of an integral by relying on the Dynkin's formula. For more details about the HJB equation and the Dynkin's formula, see Appendix B. According to the Dynkin's formula the mean value J_T, given in (9), reads, for $X_{t-s}^{s,x} \equiv X_{t-s}$

$$\begin{aligned} J(s,x;\rho) = x^2 + \mathbf{E}\Bigg[\int_s^T \Big\{[\alpha X_{t-s} - \rho_{t-s} X_{t-s}](2X_{t-s}) + \\ + \sigma^2 \rho_{t-s}^2 X_{t-s}^2\Big\}\, dt\Bigg], \end{aligned} \tag{13}$$

where T is the stopping time introduced in (10). By virtue of the invariance of the problem under time translation (that is, the homogeneous problem over time), we can rewrite $J(s,x;\rho)$, given in (13), in the form:

$$J(s,x;\rho) = x^2 + \mathbf{E}\left[\int_0^{T-s} \left\{[\alpha X_t - \rho_t X_t](2X_t) + \sigma^2 \rho_t^2 X_t^2\right\} dt\right]. \tag{14}$$

where we denote the process $(X_t^{0,x})_{t \le T}$ with X_t. Then the optimization problem can be equivalently written in the form:

$$\begin{cases} \phi(s,x) := x^2 + \inf_{\rho_t \in \mathcal{A}} \mathbf{E}\left[\int_0^{T-s} \{[\alpha X_t - \rho_t X_t](2X_t) + \sigma^2 \rho_t^2 X_t^2\}\, dt\right], \\ \text{sub} \qquad dX_t = [\alpha X_t - \rho_t X_t]dt - \sigma \rho_t X_t\, dW_t, \qquad \text{with} \quad X_0 = x, \end{cases} \tag{15}$$

where the initial value $X_0 = x$ is fixed in order the equation for X_t to have an unique solution.

Note that in choosing the optimal value for ρ_t, the government will take into account two relevant elements: first, the reduction power of the primary surplus, second, the relative volatility that the fiscal action itself may transmit to the system. As a result, the optimal solution of ρ_t is expected to strike a balance between these two opposite forces, as any effort in reducing debt is conducive to uncertain feedbacks for the system.

[12]For more details about the HJB methodology, see Fleming and Soner (2006) and Stengel (1986).

By applying the HJB equation to the second term on the right hand side of the optimization problem in (15), we obtain the following variational equation for $w \equiv \rho_t$:

$$\inf_w \left\{ [2(\alpha - w) + \sigma^2 w^2] X_t^2 + \frac{\partial \phi}{\partial t} + (\alpha - w) X_t \frac{\partial \phi}{\partial x} + \frac{1}{2} \left(\sigma^2 w^2 X_t^2 \right) \frac{\partial^2 \phi}{\partial x^2} \right\} = 0. \quad (16)$$

To find an optimal control we now derive equation (16) with respect to w and for $X_t = x$ we obtain the following equation after simplification of a factor x:

$$-2x + 2\sigma^2 w x - \frac{\partial \phi}{\partial x} + \sigma^2 w x \frac{\partial^2 \phi}{\partial x^2} = 0, \quad (17)$$

from which it immediately follows:

$$w \equiv \rho_t = \frac{2x + \dfrac{\partial \phi}{\partial x}}{\sigma^2 x \left(2 + \dfrac{\partial^2 \phi}{\partial x^2} \right)}. \quad (18)$$

To find a solution for equation (18) we try with a guess function with separated variables of the following type:

$$\phi(s, x) = c\, x^2\, g(s), \quad (19)$$

where $g(0)$ is a constant, that is $g(0) = K$, because the value function $\phi(s, x)$ in the optimization problem (15), as we shall see, is a multiple of x^2, that is it yields $\phi(0, x) = \tilde{K} x^2$. [13].

By substituting (19) into (18), where $\partial \phi / \partial x = 2cxg(s)$ and $\partial^2 \phi / \partial x^2 = 2cg(s)$, we obtain:

$$\hat{\rho}_t = \hat{\rho}(t, X_t(\omega)) = \hat{\rho} = \frac{1}{\sigma^2}, \quad (20)$$

that is the optimal Markov control $\hat{\rho}(t, X)$ is constant.[14]

From equation (20) two remarks are in order. First, the optimal correction factor is independent of the debt ratio. This implies that at the optimum the relationship between the surplus ratio and the debt-output ratio will be linear.

Second, when the coefficient diffusing uncertainty is high, the correction factor should correspondingly be low. The idea behind this result is that large shocks can potentially undermine or magnify the effectiveness of the fiscal effort, so that it is 'optimal' to limit the magnitude of the correction mechanism itself.[15]

[13]Note that HJB equation is a parabolic equation, and then the condition $\phi(0, x) = \tilde{K} x^2$ is the initial value, but we can not impose any boundary conditions because x is already fixed by the initial condition of the evolution equation in (15)

[14]Note that the constant control (20) is admissible as it belongs to the set $\mathcal{A}$ as in (A.2). Indeed a constant function is a Borel-measurable function ρ belonging to the set in (A.2). See Appendix A. Sufficient conditions ensuring that the optimization problem (15) satisfies the requirements for the optimality of the optimal Markov control solution are in Appendix C where we show that the application which associates $cx^2 g(s)$ with every pair (s, x) satisfies all the conditions of the *Verification Theorem.*

[15]To find an explicit value for the value function see Appendix C.

If we insert the optimal control (20) into the evolution equation in (7), this equation becomes

$$dX_t = \left(\alpha - \frac{1}{\sigma^2}\right) X_t \, dt - \frac{1}{\sigma} X_t \, dW_t \, , \tag{21}$$

whose solution, by virtue of Itô's lemma, is

$$X_t = x \, e^{\left(\alpha - \frac{3}{2\sigma^2}\right)t} e^{-\frac{1}{\sigma}W_t}. \tag{22}$$

The process (22) induces the following probability law:

$$\begin{aligned} \mathcal{P}(X_t < z) = F(z,t) = \mathcal{P}\left[x \, e^{\left(\alpha - 3/(2\sigma^2)\right)t} e^{-W_t/\sigma}\right] = \\ = \frac{1}{\sqrt{2\pi t}} \int_{-\sigma \log(z/a)}^{+\infty} e^{-w^2/(2t)} dw, \end{aligned} \tag{23}$$

where $a = xe^{\left(\alpha - 3/(2\sigma^2)\right)t}$.

Given the presence of uncertainty the following questions arise. What is the time needed to meet a fiscal target? How robust is the adjustment rule to adverse shocks? Or better, what is the probability that given the materialization of adverse shocks public debt is still on the right track towards a preset fiscal goal? In the next section we will address these questions proposing different approaches.

4 Measures of Fiscal Soundness

In this section we assess the properties of the debt dynamics under the optimal policy (20) proposing different measures of fiscal soundness. Specifically, we will first look at the time necessary to reach an expected fiscal consolidation objective and then we will compute the probability of reaching a fiscal objective in a specific time horizon. These two measures provide us with different information about the ability of the government to meet its fiscal consolidation target in a stochastic environment from a medium- and long-run perspective. In particular, the first measure pinpoints the time needed to reach a given objective that is expected to be achieved in the presence of shocks. This may be seen as the time resistance towards the objective when the system is placed under pressure. The second measure refers to the probability of reaching a fiscal consolidation target in a given time horizon. A higher probability is clearly the sign of a sound fiscal policy.

Finally, we will propose a measure of fiscal soundness based on the probability that the debt ratio, departing from a given interval because of shocks, tends to decline. Thus, high probability conditional to the exit of the debt ratio of a predetermined interval may be interpreted as the confidence to absorb adverse shocks that may undermine fiscal stability. From this point of view this measure of fiscal resilience may be used to asses the health of public finances in the short run.

4.1 Time Needed to Reduce the Debt Ratio

What is the time needed to reduce the debt ratio by a given amount? How do uncertainty and fundamentals affect the time required to meet a given target? These questions are relevant issues for policymakers, since the achievement of their goals is conditioned by the expected time necessary to meet them. Indeed, adverse cyclical factors and changing political conditions may considerably expand the time eventually needed to reach a given objective, prolonging the time span of a policy action. The credibility of any fiscal reform also depends on the expected time necessary to reach an established aim. The more distant in the future the achievement of the final goal, the less credible the policy action will be.

From (22), recalling (8), the expected value of the debt ratio $E[X_t]$ is

$$\mathbf{E}[X_t] = x\, e^{\left(\alpha - \frac{3}{2\sigma^2}\right)t} \int_{\mathbb{R}} e^{-w/\sigma} \frac{e^{-w^2/(2t)}}{\sqrt{2\pi t}}\, dw = x\, e^{\left(\alpha - 1/\sigma^2\right)t}, \tag{24}$$

while the variance is

$$\mathbf{E}[X_t - \mathbf{E}[X_t]]^2 = x^2\, (e^{\frac{1}{\sigma^2}t} - 1) e^{2\left(\alpha - \frac{1}{\sigma^2}\right)t}. \tag{25}$$

From (24) the expected value of the debt ratio declines over time provided that condition $1/\sigma^2 > \alpha$ holds. The speed of convergence towards an expected target is clearly increasing in α and in σ. If $1/\sigma^2 > \alpha$, then the variance of X_t will display a hump-shaped dynamics over time.

To better illustrate the behaviour of moments as time changes we will make use of a numerical example. Figure 1 presents the expected value of the debt ratio $E(X_t)$ and its standard deviation σ_{X_t}, given an initial debt ratio $x = 100\%$, $\alpha = 0.02$ under three different values of σ. Higher uncertainty will expand the time necessary to reach the objective as a result of the fact that the government will find it 'optimal' to slow down the fiscal effort in response to the higher unpredictability of the final outcome of the policy intervention. As an example, after 10 years for $\sigma = 5$, $E(X_t)$ will be about 20 p.p. higher than what observed under lower uncertainty, that is for $\sigma = 3$. After 40 years $E(X_t)$ is close to zero with $\sigma = 3$, while for $\sigma = 5$ the expected value is still more than 40 p.p. above. However, a strong reaction to the debt ratio initially generates a high variability especially when σ is lower. This is because the optimal rule (20) prescribes a strong reaction to the debt ratio when σ is low, so inducing a substantial feedback effect on α_t (see eq. (5)). At later stages, instead, the standard deviation declines faster the lower the degree of uncertainty. As long as the debt ratio declines, the amount of uncertainty is sharply reduced. This is the result of the initial trade-off faced by the policymaker at the earlier stages of the adjustment towards targeted debt reduction, discussed in Section 2.

Figure 2 shows the role played by market fundamentals in determining the time path of the expected value of the debt ratio and of its standard deviation. We observe that, other things being equal, the higher α, reflecting weak economic fundamentals, such as structural low growth and/or high interest rate, the slower the convergence towards the objective, and thus more the time needed to meet the established objective, and the higher the variability. Hence, as expected, a high α severely reduces the stabilizing properties of the rule. Overall the effects of changes in market fundamentals are magnified in the presence of high uncertainty. This can by easily explained by close inspection of equation (21), where for an increase in σ the role of market fundamentals becomes pivotal in shaping the time path of the debt ratio. In the presence of high uncertainty the optimal rule, in fact, implies a weaker reaction, so that the adjustment of the debt ratio towards a given target relies on market fundamentals at a greater extent.

Table 1 summarizes the above findings presenting the time needed to reach different debt ratio that are expected to be achieved in the presence of shocks for different values of σ and α. We consider four different expected fiscal consolidation goals and from (24) we compute the time needed to reach the objective. As before, the initial value x is set at 100%. In the more favourable scenario, with α set at 0.01 and σ at 3, the time needed to reduce the debt by 10 p.p. is 1 year, while in the worst scenario, with α at 0.03 and σ at 5, is 11 years. Similarly, reducing the debt by 40 p.p. is feasible in 5 years under favourable conditions and in 51 years under adverse circumstances. It should be noted that for low uncertainty the number of years necessary to reduce the debt only marginally depends on the size of α. For low uncertainty, in fact, the optimal rule dictates a stronger adjustment.

4.2 Probability of Reaching a Fiscal Objective

We now show how uncertainty and fundamentals affect the probability of reaching a fiscal target. Since the debt ratio is constantly bounced around by a number of shocks, the reduction of debt towards a target is uncertain. Indeed, debt trajectories are surrounded by uncertainty, as a result of two opposing forces, the correction rule pushing debt down on the one hand, and adverse shocks that may drive debt up on the other hand. Therefore, in a stochastic environment it becomes relevant for policymakers to measure the degree of confidence associated with the effectiveness of the fiscal action at play, namely the capability of pursuing an objective. To this end we use (22) to compute the probability that the debt ratio reaches specific targets after 5, 10 and 20 years under different parametrizations for σ and α. The initial value of debt ratio x is set at 100%. By using the probability law (23) induced by the process X_t, we can calculate the probability that the public debt is lower than some fixed values, as it is summarized in Table 2. As expected, adverse fundamentals (i.e. high values for α) and high uncertainty undermine fiscal sustainability.

As an example, the probability of closing the debt by 20 p.p. in 10 years (namely the case $X < 80$), corresponding to an average yearly reduction of 2 p.p. over 10 years, is around 90% under favorable GDP-growth-interest-rate conditions ($\alpha = 0.01$) and in the case that the coefficient diffusing uncertainty is low ($\sigma = 3$). In this situation, the stabilising effect of the optimal rule tends to prevail over the adverse feedback effects on interest-growth differential so as to make this objective confidently reachable. As the economic fundamentals weaken (that is α increases) the corresponding probability to meet this target falls, highlighting how the outcome of the fiscal consolidation at stake is affected by long-term/structural components of the economic system. We observe that this measure of probability drops dramatically as σ increases. In this case there are two concurring forces leaning against the reaching of a fiscal target. On the one hand a high σ entails a lower fiscal effort so as to drag down the probability of meeting the target; on the other hand a high σ increases the uncertainty of the fiscal intervention. As a result, the probability of closing the debt by 20 p.p. in 10 years when σ doubles ($\sigma = 6$) is now around 42% in the worst case scenario ($\alpha = 0.03$).

4.3 Fiscal Resilience Measure

In this section we propose a fiscal resilience measure aiming at assessing the health of public finances in a stochastic environment from a short-run perspective. In detail, we assume that the government is committed to follow the optimal rule (20). However, the stochastic component W_t in (5) may push the debt ratio up for some time, thus frustrating any policy effort and undermining fiscal stability. In this context we need to further operationalise the concept fiscal resilience choosing an appropriate indicator able to identify emerging risks.

To this end we assume that at time 0 the debt ratio x is in between two extremes, say $\underline{x}$ and $\overline{x}$, that is $x \in D := (\underline{x}, \overline{x})$. These two extremes may reflect the boundaries within which we may expect the debt ratio to vary in the short run. In a process of fiscal consolidation x will sooner or later exits this interval from below, however adverse shocks may push x to leave this interval from above at least in the short term.

We then address the following questions. What is the probability that, given uncertainty, at time t the debt ratio departing from this interval is as close as possible to $\underline{x}$? What is the role played by fundamentals and uncertainty?

In order to construct this probability we make use of the concept of harmonic measure. Formally, a harmonic measure of X_t describes its distribution as X_t hits the boundaries of D, namely $\underline{x}$ or $\overline{x}$. By virtue of the diffusion theory related to the Itô stochastic processes, we can associate the following second-order ordinary differential equation to equation (21) as follows:

$$\frac{x^2}{2\sigma^2} f''(x) + \left(\alpha - \frac{1}{\sigma^2}\right) x f'(x) = 0. \tag{26}$$

Let $f \in C^2(R)$ be a solution of this differential equation. Also, let $(\underline{x}, \overline{x}) \subset R$ be an open interval such that $x \in (\underline{x}, \overline{x})$ and put

$$\tau = \inf \left\{ t > 0 \, : \, X_t \notin (\underline{x}, \overline{x}) \right\}, \tag{27}$$

where τ measures the first instant of time in which the debt ratio does not belong to the interval $(\underline{x}, \overline{x})$.[16]

Recalling the Dynkin's formula it is possible to give a formal expression for the probability that debt is bending towards the lower bound $\underline{x}$. If $f(\overline{x}) \neq f(\underline{x})$, by

[16] We are assuming that $\tau < \infty$ almost sure with respect to the probability law Q^x by means of the Brownian motion.

using the Dynkin's formula then the probability may be written as

$$\mu^x(\underline{x}) = \frac{f(\overline{x}) - f(x)}{f(\overline{x}) - f(\underline{x})}, \tag{28}$$

that is the harmonic measure μ of X on $\underline{x}$.[17] For a formal proof of how (28) is obtained, see Appendix D. Such a harmonic measure is the probability that, in the first instant of time τ in which the process X_τ does not belong to the fixed open interval $(\overline{x}, \underline{x})$, the process assumes the value $\underline{x}$. At the operational level this interval may be anchored to the deficit limit given by the available fiscal space or set institutionally, as the deficit limit enshrined in the Maastricht Treaty.

From an economic point of view, the harmonic measure (28) may have a twofold interpretation. On the one hand, it may be seen as an index of risk about how much a system is vulnerable to potential adverse shocks. In our setup, a state jump into the wrong direction $\overline{x}$ may represent an early warning indicator of fiscal imbalances. On the other hand, $\mu^x(\underline{x})$ may be portrayed as the government ability to bend towards a target for a given α and σ, under the optimal correction rule and in the presence of external perturbations. The first interpretation highlights the capability to withstand negative shocks, while the second one refers to the capability of meeting an objective. Nevertheless, both interpretations outline the idea of resistance against adverse conditions. This is why in terms of fiscal resilience the two interpretations are interchangeable.

After some manipulations (see Appendix E for more details), it is possible to give an explicit expression for (28) as follows:

$$\mu^x(\underline{x}) = \frac{\overline{x}^{3-2\sigma^2\alpha} - x^{3-2\sigma^2\alpha}}{\overline{x}^{3-2\sigma^2\alpha} - \underline{x}^{3-2\sigma^2\alpha}}. \tag{29}$$

The value of $\mu^x(\underline{x})$ thus depends on the specific setting of α and σ. Figure 3 shows how this probability changes for different parametrization of α and σ, where we have set $x = 100\%$, $\underline{x} = 95\%$ and $\overline{x} = 105\%$. The horizontal dashed line corresponds to $\mu^x(\underline{x}) = 0.5$. The three plotted curves intersect the horizontal lines for values of σ, such that $1/\sigma^2 = \alpha$. We observe that a lower α (i.e. more favourable economic conditions) increases the probability that when departing from the interval $(\underline{x}, \overline{x})$ the debt ratio goes down. A higher probability would then signal a more resilient fiscal stance. A higher σ, instead, injects more fiscal policy uncertainty into the system so undermining fiscal stability. The lower probability would signal that the trajectory of the debt-GDP ratio could be significantly more worrisome than expected.

[17] To be sure, the corresponding harmonic measure μ of X on $\overline{x}$ can be derived as $\mu^x(\overline{x}) = 1 - \mu^x(\underline{x})$.

5 Conclusion

This paper studies the optimal debt reduction policy in a simple stochastic model of debt by using optimal control theory and applying the Hamilton-Jacobi-Bellman equation. The government is assumed to follow a simple feedback rule according to which the primary balance is adjusted to the debt-GDP ratio.

However, the government has partial control over the primary balance, since the final impact of any fiscal intervention is surrounded by uncertainty.

In such an environment the optimal Markov control policy turns out to prescribe that the reactivity of the primary balance to the debt ratio is independent of the debt ratio itself, rather it depends only on the degree of uncertainty surrounding the effects of fiscal policies on the economy. Overall, the optimal rule envisages a strong fiscal effort under more stable economic conditions. This result suggests that a simple linear rule of primary balance adjustment to the debt ratio may be optimal, provided that the size of the adjustment coefficient is tailored to the underlying market fundamentals. We propose three different measures to assess the health of fiscal finances. The first measure is simply based on the time needed to meet an expected fiscal consolidation objective. The second measure looks at the probability of reaching a fiscal target in a given period. Clearly, these two metrics can be used as to analyze fiscal soundness from a medium- and long-run perspective. The third measure, relying on harmonic measure theory, is constructed from the probability distribution of the debt ratio as it hits the boundary of a given open interval. This measure gives us the probability that the debt ratio bends towards a target and may be then interpreted as a fiscal resilience indicator that may be used to assess the stability of public finances from a short-run perspective.

As expected high growth rates and low interest rates improve the soundness of fiscal policy, while a higher degree of uncertainty jeopardizes fiscal stability. We argue that these measures could be used as simple indicators to gauge the goodness of a fiscal consolidation plan and as early warning indicators of fiscal imbalances.

In this paper we have deliberately considered a parsimonious model, yet general enough to capture several dimensions of the public debt control problem. The analysis may be extended in a number of directions. First, the analysis might be extended to account for the interaction between monetary and fiscal policies. The underlying monetary regime may facilitate or make more difficult the optimal control of public debt, and when it comes to ensure jointly price stability and fiscal solvency, the policy trade-offs may become more severe and the optimal control problem more challenging. Second, the measures of fiscal resilience proposed in this paper should be compared with other fiscal indicators and their behaviour should be analyzed in practice. We leave these aspects for future research.

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Tables and Figures

Table 1: Time Necessary to Meet an Expected Debt Ratio Target

$\sigma = 3$			
	$\alpha = 0.01$	$\alpha = 0.02$	$\alpha = 0.03$
$E(X_t) = 90$	1	1	1
$E(X_t) = 80$	2	2	3
$E(X_t) = 70$	4	4	4
$E(X_t) = 60$	5	6	6
$\sigma = 4$			
	$\alpha = 0.01$	$\alpha = 0.02$	$\alpha = 0.03$
$E(X_t) = 90$	2	2	3
$E(X_t) = 80$	4	5	7
$E(X_t) = 70$	7	8	11
$E(X_t) = 60$	10	12	16
$\sigma = 5$			
	$\alpha = 0.01$	$\alpha = 0.02$	$\alpha = 0.03$
$E(X_t) = 90$	4	5	11
$E(X_t) = 80$	7	11	22
$E(X_t) = 70$	12	18	36
$E(X_t) = 60$	17	26	51

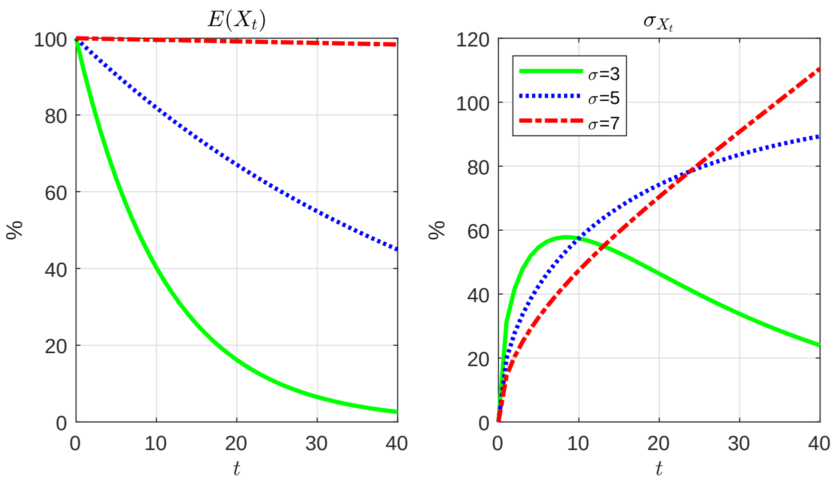
Note: the table reports the number of years necessary to meet an expected debt ratio target, expressed in %, for different combinations of uncertainty, σ, and fundamentals, α, given an initial debt ratio $x = 100\%$.

Table 2: Probability of Reaching a Debt Ratio Target

	5 years			10 years			20 years		
$\sigma = 3$									
	$\alpha = 0.01$	$\alpha = 0.02$	$\alpha = 0.03$	$\alpha = 0.01$	$\alpha = 0.02$	$\alpha = 0.03$	$\alpha = 0.01$	$\alpha = 0.02$	$\alpha = 0.03$
$\mathcal{P}\,(X < 90)$	0.8185	0.8002	0.7810	0.9172	0.9017	0.8843	0.9789	0.9711	0.9610
$\mathcal{P}\,(X < 80)$	0.7738	0.7532	0.7315	0.8988	0.8809	0.8610	0.9745	0.9655	0.9539
$\mathcal{P}\,(X < 70)$	0.7165	0.6933	06694	0.8745	0.8538	0.8310	0.9687	0.9580	0.9446
$\mathcal{P}\,(X < 60)$	0.6427	0.6173	0.5915	0.8417	0.8177	0.7916	0.9607	0.9479	0.9320
$\sigma = 4$									
	α=0.01	α=0.02	α=0.03	$\alpha = 0.01$	$\alpha = 0.02$	$\alpha = 0.03$	$\alpha = 0.01$	$\alpha = 0.02$	$\alpha = 0.03$
$\mathcal{P}\,(X < 90)$	0.7125	0.6812	0.6487	0.8228	0.7880	0.7496	0.9198	0.8897	0.8523
$\mathcal{P}\,(X < 80)$	0.6368	0.6027	0.5679	0.7815	0.7424	0.6999	0.9030	0.8686	0.8266
$\mathcal{P}\,(X < 70)$	0.5442	0.5086	0.4730	0.7285	0.6850	0.6388	0.8808	0.8414	0.7943
$\mathcal{P}\,(X < 60)$	0.4346	0.3997	0.3656	0.6603	0.6128	0.5637	0.8511	0.8058	0.7529
$\sigma = 5$									
	α=0.01	α=0.02	α=0.03	$\alpha = 0.01$	$\alpha = 0.02$	$\alpha = 0.03$	$\alpha = 0.01$	$\alpha = 0.02$	$\alpha = 0.03$
$\mathcal{P}\,(X < 90)$	0.6268	0.5838	0.5398	0.7337	0.6793	0.6209	0.8414	0.7813	0.7099
$\mathcal{P}\,(X < 80)$	0.5239	0.4794	0.4350	0.6692	0.6101	0.5484	0.8075	0.7405	0.6632
$\mathcal{P}\,(X < 70)$	0.4057	0.3630	0.3220	0.5896	0.5273	0.4643	0.7640	0.6899	0.6072
$\mathcal{P}\,(X < 60)$	0.2799	0.2435	02099	0.4932	0.4304	0.3694	0.7078	0.6268	0.5397
$\sigma = 6$									
	α=0.01	α=0.02	α=0.03	$\alpha = 0.01$	$\alpha = 0.02$	$\alpha = 0.03$	$\alpha = 0.01$	$\alpha = 0.02$	$\alpha = 0.03$
$\mathcal{P}\,(X < 90)$	0.5565	0.5032	0.4498	0.6558	0.5836	0.5086	0.7606	0.6700	0.5682
$\mathcal{P}\,(X < 80)$	0.4310	0.3790	0.3292	0.5704	0.4951	0.4199	0.7090	0.6110	0.5055
$\mathcal{P}\,(X < 70)$	0.2973	0.2526	02117	0.4697	0.3953	0.3244	0.6447	0.5410	0.4343
$\mathcal{P}\,(X < 60)$	0.1721	0.1401	0.1123	0.3563	0.2884	0.2273	0.5653	0.4586	0.3548

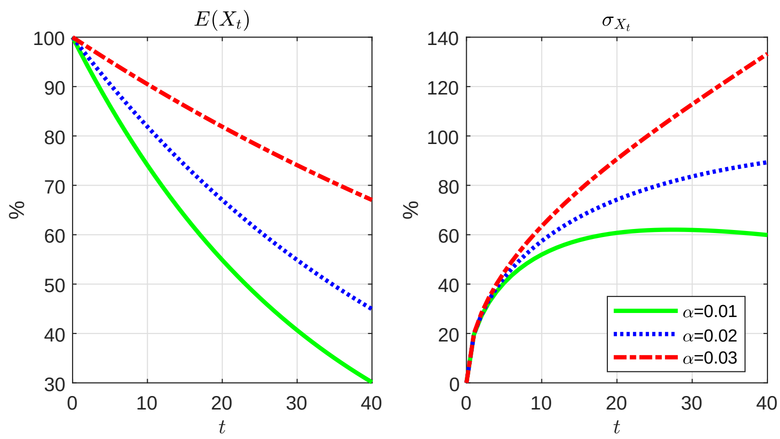
Note: the table reports the probability that debt ratio is below a certain threshold level for different combinations of uncertainty, σ, and fundamentals, α, given an initial debt ratio $x = 100\%$.

Figure 1: Debt Ratio and Uncertainty - Theoretical Moments



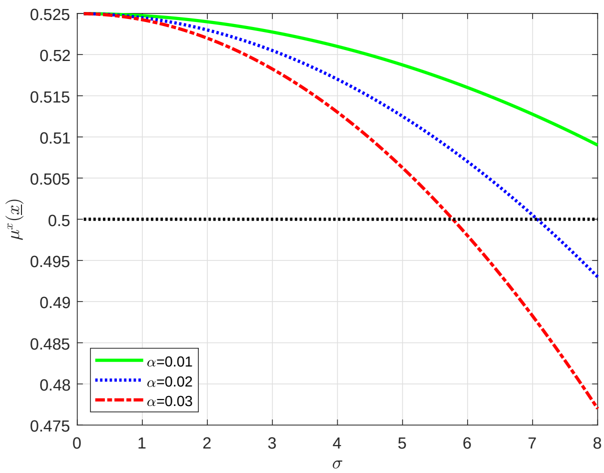
Note: the figure plots the mean and the standard deviation of the debt ratio X_t for different values of uncertainty, σ, given an initial debt ratio $x = 100\%$ and fundamentals $\alpha = 0.02$.

Figure 2: Debt Ratio and Fundamentals - Theoretical Moments



Note: the figure plots the mean and the standard deviation of the debt ratio X_t for different values of fundamentals, α, given an initial debt ratio $x = 100\%$ and uncertainty $\sigma = 5$.

Figure 3: Fiscal Resilience, Uncertainty, and Fundamentals



Note: the figure plots $\mu^x(\underline{x})$ for different values of uncertainty, σ, and fundamentals, α, given an initial debt-ratio gap $x = 100\%$ and interval boundaries $\overline{x} = 95\%, \underline{x} = 105\%$.

Appendix A

In this appendix we shortly discuss some of the mathematical features underlying equation (7). The intent is twofold: to clarify the mathematical notation of the text and define precisely the control variable.

Definition 1. Given a set Ω, a $\sigma - algebra$ $\mathcal{F}$ on Ω is a *family* $\mathcal{F}$ of subsets of Ω that fulfill the following properties:

(i) the empty set $\emptyset$ belongs to $\mathcal{F}$;

(ii) if $F \in \mathcal{F}$, then the complement $\bar{F}$ of F in Ω belongs to $\mathcal{F}$, too;

(iii) if $A_1, A_2, A_3, \dots \in \mathcal{F}$, then $A := \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$. □

Definition 2. The pair $(\Omega, \mathcal{F})$ is called a *measurable space.* □

Definition 3. A *probability measure* $\mathbb{P}$ on a measurable space $(\Omega, \mathcal{F})$ is a function $\mathbb{P} : \mathcal{F} \longrightarrow [0, 1]$ such that

(i) $\mathbb{P}(\emptyset) = 0$ and $\mathbb{P}(\Omega) = 1$;

(ii) if $A_1, A_2, A_3, \dots \in \mathcal{F}$ and $A_i \cap A_j = \emptyset$, $\forall i \neq j$, then $\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$.

□

Definition 4. The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *probability space.* It is called a *complete probability space* if $\mathcal{F}$ contains all subsets S of Ω with $\mathbb{P} -$ *outer measure* zero, where the $\mathbb{P} -$ *outer measure*, denoted by $\mathbb{P}^*$, is defined as

$$\mathbb{P}^*(G) = \inf \Big\{ \mathbb{P}(F) : F \in \mathcal{F} \text{ and } G \subset F \Big\} .$$ □

Definition 5. For a given *family* $\mathcal{G}$ of subsets of Ω, the $\sigma - algebra$ denoted by the symbol $\mathcal{F}_{\mathcal{G}}$ and defined as

$$\mathcal{F}_{\mathcal{G}} = \bigcap \Big\{ \mathcal{F} \, : \, \mathcal{F} \text{ is a } \sigma - algebra \text{ of } \Omega \text{ and } \mathcal{G} \subset \mathcal{F} \Big\}$$

is called the $\sigma - algebra$ *generated by* $\mathcal{G}$. □

Definition 6. If Ω is a topological space (e.g. $\Omega = \mathbb{R}^n$) equipped with the topology $\mathcal{G}$ of all open subsets of Ω, then the $\sigma - algebra$ $\mathcal{B} = \mathcal{F}_{\mathcal{G}}$ is called the *Borel* $\sigma - algebra$ on Ω and the elements $B \in \mathcal{B}$ are called *Borel sets.* □

Definition 7. Given the *measurable space* $(\Omega, \mathcal{F})$, the (increasing) family $\{\mathcal{M}_t\}_{t \geq 0}$ of $\sigma - algebras$ of Ω such that

$$\mathcal{M}_{t_1} \subset \mathcal{M}_{t_2} \subset \mathcal{F} \, , \qquad \forall \; 0 \leq t_1 < t_2 \, ,$$

is called a *filtration* on $(\Omega, \mathcal{F})$. □

Since for every t we have a random control variable which the random variable X_t depends upon, we consider a complete probability space Ω with filtration $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, where the filtration $\mathcal{F}_t$ is the one generated by the standard one-dimensional Brownian motion W_t and is augmented by $\mathbb{P}$-null sets, that is

$$\mathcal{F}_t = \sigma\Big(W_s,\ \ 0 \leq s \leq t\Big) \cup \Big\{A \in \mathcal{F} \,|\, \mathbb{P}(A) = 0\Big\}, \qquad \forall\ t \geq 0. \tag{A.1}$$

Moreover, we assume that X_0 is an integrable random variable with law π_0 and measurable with respect to $\mathcal{F}_0$ representing the initial value of the current debt-to-output ratio. For the sake of simplicity, we assume that at the time 0 the debt-ratio is deterministic, so that $X_0 = x$ is a constant with $x > 0$.

The control variable $\rho_t = \rho(t, \omega)$ is taken from a given family $\mathcal{A}$ of admissible controls:

$$\begin{aligned} \mathcal{A} := \Big\{\rho_t = \rho(t, X_t(\omega)) \text{ for some Borel-measurable} \\ \text{bounded functions } \rho : [0, T] \times [0, \bar{x}] \longrightarrow [0, \bar{\rho}]\Big\} \end{aligned} \tag{A.2}$$

and is adapted to the filtration (A.1).

At time t the value of the function $\rho(t, X_t)$ only depends on the state of the system at that time, thus it does not depend on the probability space ω explicitly, but only through the process X_t. Such a ρ is called *Markov control* and the corresponding process X_t becomes an Itô diffusion, in particular a Markov process (see Øksendal 2003, Section 11.1).

Appendix B

Remark 1. *There exists a unique solution for the controlled equation (11) (see Øksendal (2003) for more details).*

For $\bar{X} < X_f$ let us define the exit time τ of the dynamics as

$$\tau := \inf\{t > 0 \mid X_t > \bar{X}\}. \tag{B.1}$$

Remark 2. *By virtue of well known results, the measurability of τ with respect to the σ-algebra $\mathcal{F}_t$ follows. Indeed, we have that τ is a stopping time.*

Theorem 1 (Dynkin's formula). *Let X_t be the Itô diffusion*

$$dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dB_t, \quad X_0 = y, \tag{B.2}$$

and $f \in C_0^2(\mathbb{R})$. If τ is a stopping time with $\mathbf{E}[\tau] < +\infty$, then the following relationship holds:

$$\mathbf{E}[f(X_\tau)] = f(x) + \mathbf{E}\left[\int_0^\tau Lf(X_s)\, ds\right], \tag{B.3}$$

where

$$Lf(z) := \mu(z)\frac{df}{dz} + \frac{1}{2}[\sigma(z)]^2 \frac{d^2 f}{dz^2}. \tag{B.4}$$

Theorem 2 (HJB equation). *Suppose that we have*

$$V(s,y) := \sup_{\rho_t \in \mathcal{A}} \mathbf{E}\left[\int_s^\tau f(X_t, \rho_t)\, dt\right], \tag{B.5}$$

with

$$\begin{cases} dX_t = \mu(X_t,\ \rho_t)\, dt + \sigma(X_t,\ \rho_t)\, dB_t, \\ X_0 = x\ . \end{cases} \tag{B.6}$$

Suppose that $V \in C^2(\mathbb{R}^+)$ satisfies

$$\mathbf{E}\left[|V(X_\alpha)| + \int_0^\alpha |L^\rho V(X_t)|\, dt\right] < +\infty, \tag{B.7}$$

for all bounded stopping times $\alpha < \tau$, for all $x \in \mathbb{R}$ and all $\rho \in \mathcal{A}$, where

$$(L^z V)(s,x) := \frac{\partial V(s,x)}{\partial s} + \mu(x,\rho)\frac{\partial V}{\partial x} + \frac{\sigma^2(x,\rho)}{2}\frac{\partial^2 V}{\partial x^2}. \tag{B.8}$$

Moreover, suppose that an optimal control ρ^* exists, then we have

$$\sup_{\rho \in \mathcal{A}}\{f(x,\rho) + (L^\rho V)(x)\} = 0, \tag{B.9}$$

and the supremum is obtained if it yields $\rho = \rho_t^*$, that is

$$f(x, \rho^*(t)) + (L^{\rho^*(t)} V)(x) = 0\,. \tag{B.10}$$

Theorem 2 also applies to the corresponding *minimum* problem

$$\phi(s,x) := \inf_{\rho_t \in \mathcal{A}} \mathbf{E}\left[\int_s^\tau f(X_t,\rho_t)\,dt\right]. \tag{B.11}$$

We have in fact

$$\phi(s,x) = -\sup_{\rho_t \in \mathcal{A}} \mathbf{E}\left[\int_s^\tau -f(X_t,\rho_t)\,dt\right],$$

from which, by replacing V with $-\phi$ and f with $-f$, it follows that the (B.9) in Theorem 2 becomes:

$$\inf_{\rho \in \mathcal{A}} \{f(x,v) + (L^z\phi)(x)\} = 0. \tag{B.12}$$

For the details the reader may refer to Øksendal (2003).

Appendix C

In this appendix we give sufficient conditions to conclude that (20) is the optimal Markov control process and how the corresponding value function is. The proof relies essentially on Itô's lemma as follows.

First we give and explicit expression of the value function $\phi(s,x)$ **.** To find it we have to substitute the expressions (19) and (20) into equation (16) in order to obtain the following separated equation for the temporal function $g(s)$:

$$g'(s) + \left(2\alpha - \frac{1}{\sigma^2}\right) g(s) = \frac{1}{c}\left(\frac{1}{\sigma^2} - 2\alpha\right). \tag{C.1}$$

If we substitute the solution (22) into the integral in (15), the value function, for $s = 0$, becomes:

$$\phi(0,x) = x^2 + (2\alpha - 1/\sigma^2)x^2 \ \mathbf{E}\left[\int_0^T e^{(2\alpha - 3/\sigma^2)t - 2W_t/\sigma} dt\right] = \tilde{K}x^2,$$

because the mean value of a random variable is obviously a constant.

If we now consider the initial condition $g(0) = K \equiv \tilde{K}/c$, the solution of the temporal ordinary differential equation (C.1) reads:

$$g(s) = \left(K + \frac{1}{c}\right) e^{\left(1/\sigma^2 - 2\alpha\right)s} - \frac{1}{c}. \tag{C.2}$$

We then have an explicit expression of the value function $\phi(s,x)$ given by $\phi(s,x) = \left[(Kc+1)\, e^{\left(1/\sigma^2 - 2\alpha\right)s} - 1\right] x^2$.

Verification Theorem

Let

$$\phi(s,x) := \inf_{\rho_t \in \mathcal{A}} \mathbf{E} \int_0^{T-s} f(t, X_t, \rho_t)\, dt,$$

with

$$dX_s = b(s, X_s, \rho(s, X_s))ds + \sigma(s, X_s, \rho(s, X_s))dW_s \quad \text{and} \quad X_0 = x,$$

be an optimisation problem. Let V be a $C^{1,2}([0,T) \times \mathbb{R}) \cap C([0,T) \times \mathbb{R})$ function and let us assume that f and V have quadratic growth, i.e. there is a constant C such that

$$|f(t,x,\rho)| + |V(t,x)| \leqslant C(|x|^2 + 1), \tag{C.3}$$

for all $(t,x,\rho) \in [0,T) \times \mathbb{R} \times \mathcal{A}$. (i) Suppose that

$$\frac{\partial V(t,x)}{\partial t} + f(t,x,\rho) + b(t,x,\rho)\,\frac{\partial V(t,x)}{\partial x} + \frac{\sigma^2(t,x,\rho)}{2}\,\frac{\partial^2 V(t,x)}{\partial x^2} \geqslant 0 \tag{C.4}$$

on $[0,T) \times \mathbb{R}$. Then $V \leqslant \phi$ on $[0,T] \times \mathbb{R}$. (ii) Assume further that there exists a minimizer $\hat{\rho}(t,x)$ of the function

$$\rho \to \mathcal{L}^\rho V(t,x) + f(t,x,\rho),$$

such that

$$0 = \frac{\partial V(t,x)}{\partial t} + \inf_{\rho \in \mathcal{A}} \left\{ f(t,x,\rho) + b(t,x,\rho)\, \frac{\partial V(t,x)}{\partial x} + \frac{\sigma^2(t,x,\rho)}{2}\, \frac{\partial^2 V(t,x)}{\partial x^2} \right\} =$$
$$= \frac{\partial V(t,x)}{\partial t} + \mathcal{L}^{\hat{\rho}(t,x)} V(t,x) + f(t,x,\hat{\rho}), \tag{C.5}$$

where $\mathcal{L}^{\rho} V(t,x)$ is defined as

$$\mathcal{L}^{\rho} V(t,x) := b(t,x,\rho)\, \frac{\partial V(t,x)}{\partial x} + \frac{\sigma^2(t,x,\rho)}{2}\, \frac{\partial^2 V(t,x)}{\partial x^2}\,. \tag{C.6}$$

Then the stochastic differential equation

$$dX_s = b(s, X_s, \hat{\rho}(s, X_s))ds + \sigma(s, X_s, \hat{\rho}(s, X_s))dW_s \tag{C.7}$$

defines a unique solution X for each given initial date $X_0 = x$ and the process $\hat{\rho} := \hat{\rho}(s, X_s)$ is a well-defined control process in $\mathcal{A}$. Then ϕ is the value function and $\hat{\rho}$ is the optimal Markov control process. In our case from equation (15) we have

$$f(t,x,\rho) = \left[\sigma^2(1+\alpha)^2\rho^2 + 2\alpha - 2\rho(1+\alpha)\right] x^2, \tag{C.8}$$

which has quadratic growth and then it follows that there exists a positive constant such that

$$C_1 \geqslant \sup_{\rho} \left\{ |\sigma^2(1+\alpha)^2\rho^2 + 2\alpha - 2\rho(1+\alpha)| \right\} \tag{C.9}$$

and $|f(t,x,\rho)| \leqslant C_1 x^2$. Since the term in square brackets in ((C.8)) is bounded, it follows that there exists a positive constant C_2 such that $|f(t,x,\rho)| \leqslant C_1 x^2$. Then, the condition (C.3) is satisfied with a positive constant $C > C_1 + C_2 - 1$. Further, the condition (C.4) is verified, too, because the expression

$$\frac{\partial V(t,x)}{\partial t} + f(t,x,\rho) + b(t,x,\rho)\, \frac{\partial V(t,x)}{\partial x} + \frac{\sigma^2(t,x,\rho)}{2}\, \frac{\partial^2 V(t,x)}{\partial x^2} \tag{C.10}$$

is convex with respect to ρ and thus positive or null for each $\rho \neq \hat{\rho}$. Then, according to the Verification Theorem, the value function is thus $\phi(s,x)$, and the optimal policy is the constant process $\hat{\rho}_t$.

Appendix D

In this appendix we derive equation (28) by means of Dynkin's formula. Specifically, let

$$dX_t = r(X_t)dt + \sigma(X_t)dB_t \tag{D.1}$$

be a 1-dimensional Itô diffusion with characteristic operator $\mathcal{A}$ and $f \in C^2(\mathbb{R})$ be a solution of the ordinary differential equation:

$$\mathcal{A}f(x) = r(x)f'(x) + \frac{\sigma^2(x)}{2} f''(x) = 0, \quad x \in \mathbb{R}. \tag{D.2}$$

Let $(a, b) \subset \mathbb{R}$, with $b > a > 0$, be an open interval such that $x \in (a, b)$ and put

$$\tau \equiv \tau(a, b) = \inf\{t > 0 \text{ such that } X_t \notin (a, b)\},$$

and assume that $\tau < \infty$ a.s. with respect to the probability law of X_t. If we define

$$p \equiv P^x[X_\tau = b],$$

it follows

$$p = \frac{f(x) - f(a)}{f(b) - f(a)}\,. \tag{D.3}$$

Proof. If we consider the function $f_0 \in C_0^2(\mathbb{R})$ such that $f_0(x) \equiv f(x)$ on (a, b) and $\mathcal{A}f_0(x) = \mathcal{A}f(x) = 0$, by means of Dynkin formula we can write:

$$E^x[f(X_\tau)] = f(x) + E^x\left[\int_0^\tau \mathcal{A}f(X_s)\,ds\right] = f_0(x). \tag{D.4}$$

Since $f_0(x) \in C_0^2(\mathbb{R})$ and $X_{\tau(a,b)} \notin (a, b)$, it follows that the random variable $X_{\tau(a,b)}$ can assume the two values a and b, only. Then the mean value $E^x[f(X_\tau)]$ of $f(X_\tau)$ is given by the sum of the two products of the values $f_0(a)$ and $f_0(b)$ multiplied by the corresponding probabilities, $1 - p$ and p respectively, that is

$$f_0(x) \equiv E^x[f(X_\tau)] = f_0(a)(1 - p) + f_0(b)p. \tag{D.5}$$

From the equality between the first and the third term we obtain the final relation

$$p(b) = \frac{f(x) - f(a)}{f(b) - f(a)}, \tag{D.6}$$

and thus:

$$p(a) = \frac{f(b) - f(x)}{f(b) - f(a)}, \tag{D.7}$$

because the equalities on the boundary of the interval $f_0(a) = f(a)$ and $f_0(b) = f(b)$ hold. In the text we assume that $a = \underline{x}$ and $b = \overline{x}$ from which it follows equation (28).

Appendix E

In order to give an explicit expression to (28) we have first to solve equation (26). To this aim we transform it into a first order differential equation through the change of the variable $f'(x) = g(x)$, so that our equation now reads:

$$\frac{x^2}{2\sigma^2}\frac{dg(x)}{dx} + \left(\alpha - \frac{1}{\sigma^2}\right) x\, g(x) = 0. \tag{E.1}$$

By separating $x, g(x)$ one obtains

$$\frac{dg}{g} = \left(2 - 2\sigma^2\alpha\right)\frac{dx}{x}, \tag{E.2}$$

whose solution is

$$g(x) = (f')^{2-2\sigma^2\alpha}\,. \tag{E.3}$$

By integration, we finally obtain the function $f(x)$:

$$f(x) = \int g(x)\,dx = C\left(\frac{x^{3-2\sigma^2\alpha}}{3-2\sigma^2\alpha}\right) + K. \tag{E.4}$$

We now have an explicit expression of the harmonic measure that debt hits the extremes of D as follows:

$$\mu^x(\underline{x}) = \frac{\overline{x}^{3-2\sigma^2\alpha} - x^{3-2\sigma^2\alpha}}{\overline{x}^{3-2\sigma^2\alpha} - \underline{x}^{3-2\sigma^2\alpha}}, \tag{E.5}$$

$$\mu^x(\overline{x}) = 1 - \mu^x(\underline{x}). \tag{E.6}$$