Inflation Shocks and Interest Rate Rules

Technical Appendix*

Barbara Annicchiarico† Alessandro Piergallini‡

A. Consumer's Intertemporal Optimization

The solution to the consumer's intertemporal maximizing problem yields the following first order necessary conditions:

$$1 = \beta R_t E_t \left\{ \frac{C_{s,t}(j) - V(N_{s,t}(j))}{C_{s,t+1}(j) - V(N_{s,t+1}(j))} \frac{P_t}{P_{t+1}} \right\}, \tag{1A}$$

$$\frac{W_{s,t}(j)}{P_t} = (1 + u_t^w) V'(N_{s,t}(j)), \tag{2A}$$

where (1A) is the stochastic Euler equation and (2A) is the efficiency condition on labor supply, featuring the exogenous optimal wage markup $u_t^w = 1/(\eta_t - 1)$. In the symmetric equilibrium workers of all generations will set the same wage and supply the same hours of labor, i.e. $W_{s,t}(j) = W_t$ and $N_{s,t}(j) = N_t$ for all $j \in [0, 1]$.

B. Individual Consumption

Define the stochastic discount factor of the representative agent j of generation s as

$$Q_{t,t+1}(s, j) \equiv \beta \frac{\Omega_{s,t}(j)}{\Omega_{s,t+1}(j)} \frac{P_t}{P_{t+1}}, \tag{1B}$$

*This Technical Appendix is not for publishing.

†Department of Economics, University of Rome 'Tor Vergata', Via Columbia 2, 00133 Roma, Italy. E-mail: barbara.annicchiarico@uniroma2.it

‡Department of Financial and Management Studies (DeFiMS), SOAS, University of London, Thornhaugh Street, London, WC1H 0XG, United Kingdom. E-mail: Ap50@soas.ac.uk

where $\Omega_{s,t}(j) \equiv [C_{s,t}(j) - V(N_{s,t}(j))]$ is the sub-utility function, that can be interpreted as consumption net of its subsistence level (see Ascari and Rankin, 2006), and $E_t\{Q_{t,t+1}(s,j)\} = 1/R_t$ for each $s \in (-\infty, t]$. At the optimum the flow budget constraint (2) holds with equality and the transversality condition, $\lim_{T\to\infty} E_t\left\{Q_{t,T}(s,j)(1-\gamma)^{T-t} B_{s,T}(j)\right\} = 0$, where $Q_{t,T}(s,j) \equiv \prod_{k=t+1}^{T} Q_{k-1,k}(s,j)$ and $Q_{t,t}(s,j) \equiv 1$, must hold.

Using the fact that $E_t\{Q_{t,t+1}(s,j)\} = 1/R_t$, we can write

$$\frac{B_{s,t+1}(j)}{R_t} = E_t\{Q_{t,t+1}(s,j)B_{s,t+1}(j)\}. \tag{2B}$$

Thus, the individual flow budget constraint (2) (which at optimum holds with equality) takes the following form:

$$(1-\gamma)E_t\{Q_{t,t+1}(s,j)\,B_{s,t+1}(j)\} = B_{s,t}(j) + \overline{Y}_{s,t}(j) - P_t\Omega_{s,t}(j) - P_tV(N_{s,t}(j)), \tag{3B}$$

where $\overline{Y}_{s,t}(j) = W_{s,t}(j)\,N_{s,t}(j) + Z_{s,t}(j) - T_{s,t}(j)$. From (3B), imposing the transversality condition, we obtain the intertemporal budget constraint:

$$\begin{aligned} & E_t\sum_{T=t}^{\infty} Q_{t,T}(s,j)(1-\gamma)^{T-t}P_T\Omega_{s,T}(j) \\ = \; & B_{s,t} + E_t\sum_{T=t}^{\infty} Q_{t,T}(s,j)(1-\gamma)^{T-t}\overline{Y}_{s,t}(j) + \\ & -E_t\sum_{T=t}^{\infty} Q_{t,T}(s,j)(1-\gamma)^{T-t}P_TV(N_{s,T}(j)). \end{aligned} \tag{4B}$$

From (1B) we have

$$Q_{t,T}(s,j)\,P_T\Omega_{s,T}(j) \equiv \beta^{T-t}P_t\Omega_{s,t}(j), \tag{5B}$$

$$E_t\{Q_{t,T}(s,j)\,P_T\Omega_{s,T}(j)\} \equiv \beta^{T-t}P_t\Omega_{s,t}(j). \tag{6B}$$

Substituting (6B) into (4B) yields the individual 'adjusted' consumption function:

$$P_t\Omega_{s,t}(j) = \Psi\left[B_{s,t}(j) + H_{s,t}(j) - E_t\sum_{T=t}^{\infty} Q_{t,T}(s,j)(1-\gamma)^{T-t}P_TV(N_{s,T}(j))\right], \tag{7B}$$

where $H_{s,t}(j) \equiv E_t\sum_{T=t}^{\infty} Q_{t,T}(s,j)(1-\gamma)^{T-t}(W_{s,T}(j)\,N_{s,T}(j) + Z_{s,T}(j) - T_{s,T}(j))$ is human wealth, defined as the expected present discounted value of future labor income and of profit shares net of taxes, and $\Psi \equiv [1-\beta(1-\gamma)]$.

C. Derivation of Equation (7)

The aggregate budget constraint (4) can be re-written as follows:

$$E_t\left\{Q_{t,t+1}B_{t+1}\right\} = B_t + \overline{Y}_t - P_t\Omega_t - P_tV\left(N_t\right). \tag{1C}$$

Solving (1C) for B_t, substituting into (5) and using the definition of aggregate human wealth, one obtains

$$\begin{aligned}\Psi^{-1}P_t\Omega_t &= E_t\left\{Q_{t,t+1}B_{t+1}\right\} + P_t\Omega_t + E_t\sum_{T=t+1}^{\infty}Q_{t,T}\left(1-\gamma\right)^{T-t}\overline{Y}_T + \\ &\quad -E_t\sum_{T=t+1}^{\infty}Q_{t,T}\left(1-\gamma\right)^{T-t}P_TV\left(N_T\right).\end{aligned} \tag{2C}$$

Leading (2C) forward one period yields

$$\begin{aligned}\Psi^{-1}P_{t+1}\Omega_{t+1} &= B_{t+1} + E_{t+1}\sum_{T=t+1}^{\infty}Q_{t+1,T}\left(1-\gamma\right)^{T-(t+1)}\overline{Y}_T + \\ &\quad -E_{t+1}\sum_{T=t+1}^{\infty}Q_{t+1,T}\left(1-\gamma\right)^{T-(t+1)}P_TV\left(N_T\right).\end{aligned} \tag{3C}$$

Multiplying both sides by $Q_{t,t+1}\left(1-\gamma\right)$ and taking expectations gives

$$\begin{aligned}\left(1-\gamma\right)\Psi^{-1}E_t\left\{Q_{t,t+1}P_{t+1}\Omega_{t+1}\right\} &= \left(1-\gamma\right)E_t\left\{Q_{t,t+1}B_{t+1}\right\} + \\ &\quad +E_t\sum_{T=t+1}^{\infty}Q_{t,T}\left(1-\gamma\right)^{T-t}\overline{Y}_T + \\ &\quad -E_t\sum_{T=t+1}^{\infty}Q_{t,T}\left(1-\gamma\right)^{T-t}P_TV\left(N_T\right).\end{aligned} \tag{4C}$$

Solving (4C) for $E_t\sum_{T=t+1}^{\infty}Q_{t,T}\left(1-\gamma\right)^{T-t}\overline{Y}_T - E_t\sum_{T=t+1}^{\infty}Q_{t,T}\left(1-\gamma\right)^{T-t}P_TV\left(N_T\right)$, substituting into (2C) and rearranging, one obtains

$$P_t\Omega_t = \frac{1}{\beta}E_t\left\{Q_{t,t+1}P_{t+1}\Omega_{t+1}\right\} + \frac{\gamma\Psi}{\beta\left(1-\gamma\right)}E_t\left\{Q_{t,t+1}B_{t+1}\right\}. \tag{5C}$$

This shows equation (7).

D. Equilibrium Adjusted Consumption and Real Marginal Costs

Using the goods market clearing condition and the aggregate production function, the equilibrium aggregate level of adjusted consumption is given by

$$\Omega_t = \left[Y_t - V(\delta_t Y_t)\right]. \tag{1D}$$

Combining the aggregate labor supply, the cost minimization condition, and the aggregate production function, one obtains the following expression for real marginal cost, MC_t:

$$MC_t = (1 + u_t^w)\, V'(\delta_t Y_t). \tag{2D}$$

E. Steady State Analysis

The steady state, around the equilibrium conditions are log-linearized, is such that $Y_t = Y > 0$, $\Omega > 0$, $P_t = P > 0$, $MC_t = MC > 0$, $R_t = R > 1$, $B > 0$, and $D_t = D > 0$ for all $t \geq 0$. This steady state is also the flexible price equilibrium[1], where

$$MC = (1 + u^w)\, V'(Y) = \frac{\varepsilon - 1}{\varepsilon}, \tag{1E}$$

$$\delta = 1. \tag{2E}$$

From equations (7) and (11) it must be that

$$R = \frac{1}{\beta} + \frac{\gamma\Psi}{\beta\,(1-\gamma)}\frac{B}{P\Omega}, \tag{3E}$$

$$\frac{B}{R} = D^n. \tag{4E}$$

From (3E) it should be noted that as long as private agents have finite horizons ($\gamma > 0$) the steady state real interest rate is affected by the steady state non-human wealth. Only

[1]This requires the standard assumption that in the flexible price equilibrium the wage markup is fixed at its steady state value $1 + u^w$ (e.g., Clarida *at al.* 2002).

in the limiting case of infinite horizon ($\gamma = 0$) the steady state real interest rate is pinned down by the subjective rate of time preference, $R = 1/\beta$.

F. Linearized Equilibrium Conditions

On the demand-side, the equilibrium adjusted consumption (1D) approximates to

$$\omega_t = \sigma y_t, \tag{1F}$$

where $\sigma \equiv [1 - V'(Y)] Y/\Omega = Y/\varepsilon\Omega$. The law of motion for ω_t can be obtained substituting (11) into (7) and log-linearizing around the steady state:

$$\omega_t = -\frac{1}{1+\lambda}\left(r_t - E_t\left\{\pi_{t+1}\right\}\right) + \frac{1}{1+\lambda}E_t\left\{\omega_{t+1}\right\} + \frac{\lambda}{1+\lambda}d_t, \tag{2F}$$

where $\lambda \equiv \gamma\Psi RD^n/\left(1-\gamma\right)P\Omega$, $\pi_t \equiv p_t - p_{t-1}$ is the inflation rate, and $d_t \equiv (d_t^n - p_t)$ is the end-of-period real public debt.

On the supply-side, log-linear approximations of the optimal price setting equation (8) and the definition of price index imply

$$\pi_t = \frac{1}{R}E_t\left\{\pi_{t+1}\right\} + \tilde{\kappa} mc_t, \tag{3F}$$

where $\tilde{\kappa} \equiv (1-\theta)(R-\theta)/R\theta$. From (2D), the log-linear version of real marginal cost is given by

$$mc_t = \eta y_t + u_t^w, \tag{4F}$$

where $\eta \equiv V''(Y)Y/V'(Y) = \varepsilon V''(Y)Y/(\varepsilon - 1)$.

Substituting (1F) into (2F) and (4F) into (3F), respectively, yields the IS equation (13) and the Phillips curve (14).

References

Ascari, G., and N. Rankin (2006) "Perpetual Youth and Endogenous Labour Supply: A Problem and a Possible Solution" *Journal of Macroeconomics*, forthcoming.

Clarida, R., J. Galí, and M. Gertler (2002) "A Simple Framework for International Monetary Policy Analysis" *Journal of Monetary Economics* **49**, 879-904.