Government Deficits, Wealth Effects and the Price Level in an Optimizing Model

TECHNICAL APPENDIX[1]

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Solution to the Consumer Optimization Problem

The representative agent of the generation born at time s face the following problem

$$\max_{\{c_{s,t}\}_{t=s}^{\infty}\{m_{s,t}\}_{t=s}^{\infty}} \sum_{t=1}^{\infty}(\beta\gamma)^t \log[(c_{s,t})^{\xi}(m_{s,t})^{1-\xi}], \tag{1T}$$

subject to

$$c_{s,t}+a_{s,t}+m_{s,t}=w_{s,t}-\tau_{s,t}+\frac{1}{\gamma}\left[\frac{m_{s,t-1}}{1+\pi_t}+(1+r_{t-1})a_{s,t-1}\right], \tag{2T}$$

where all the variables are defined as in the main text. Define $\omega_{s,t}$ the amount of resources available at time t

$$\omega_{s,t}\equiv c_{s,t}+a_{s,t}+m_{s,t}=w_{s,t}-\tau_{s,t}+\frac{1}{\gamma}\left[\frac{m_{s,t-1}}{1+\pi_t}+(1+r_{t-1})a_{s,t-1}\right]. \tag{3T}$$

The initial level of resources is the state variable for the consumer's optimization problem. The non-monetary wealth can be expressed in terms of total resources available at time t as follows

$$a_{s,t}=\omega_{s,t}-c_{s,t}-m_{s,t}. \tag{4T}$$

Updating equation (3T) of one period given (4T) yields

$$\omega_{s,t+1}\equiv w_{s,t+1}-\tau_{s,t+1}+\frac{1}{\gamma}\left[\frac{m_{s,t}}{1+\pi_{t+1}}+(1+r_t)(\omega_{s,t}-c_{s,t}-m_{s,t})\right]. \tag{5T}$$

The value function $V(\omega_{s,t})$ is given by

$$V(\omega_{s,t})=\max\left[U(c_{s,t},m_{s,t})+\beta\gamma V(\omega_{s,t+1})\right], \tag{6T}$$

where $V(\bullet)$ is the value function and $U(c_{s,t},m_{s,t})=\log[(c_{s,t})^{\xi}(m_{s,t})^{1-\xi}]$. Recalling the definition of $\omega_{s,t+1}$, the first order conditions for this optimization problem are

$$U_c(c_{s,t},m_{s,t})-\beta(1+r_t)V_{\omega}(\omega_{s,t+1})=0, \tag{7T}$$

[1]This Technical Appendix is not for publishing.

$$U_m(c_{s,t},m_{s,t})-\beta V_\omega(\omega_{s,t+1})\left[\frac{1}{1+\pi_{t+1}}-(1+r_t)\right]=0. \tag{8T}$$

The envelope theorem implies that $U_c(c_{s,t},m_{s,t})=V_\omega(\omega_{s,t})$. The first order conditions can be re-written as

$$U_c(c_{s,t},m_{s,t})-\beta(1+r_t)U_c(c_{s,t+1},m_{s,t+1})=0, \tag{9T}$$

and

$$U_c(c_{s,t+1},m_{s,t+1})=\frac{1+i_t}{i_t}U_m(c_{s,t+1},m_{s,t+1}). \tag{10T}$$

Given the utility function $U(c_{s,t},m_{s,t})=\log[(c_{s,t})^{\xi}(m_{s,t})^{1-\xi}]$, conditions (9T) and (10T) can be re-written as

$$c_{s,t+1}-\beta(1+r_t)c_{s,t}=0, \tag{11T}$$

$$\eta c_{s,t}=\frac{i_t}{1+i_t}m_{s,t}, \tag{12T}$$

where $\eta\equiv\frac{1-\xi}{\xi}$.

Derivation of the individual consumption function

Re-write the individual's budget (2T) constraint as follows

$$a_{s,t}+m_{s,t}=\tilde{\omega}_{s,t}-c_{s,t}+\frac{1}{\gamma}\frac{m_{s,t-1}}{1+\pi_t}+\frac{1}{\gamma}(1+r_{t-1})a_{s,t-1}, \tag{13T}$$

where $\tilde{\omega}_{s,t}=\omega_{s,t}-\tau_{s,t}$.

Update equation (13T) of one period and divide both sides by $(1+r_t)$

$$\frac{a_{s,t+1}}{1+r_t}+\frac{m_{s,t+1}}{1+r_t}=\frac{\tilde{\omega}_{s,t+1}-c_{s,t+1}}{1+r_t}+\frac{1}{\gamma}\frac{m_{s,t}}{(1+\pi_{t+1})(1+r_t)}+\frac{1}{\gamma}a_{s,t}, \tag{14T}$$

This last expression can be re-written as follows

$$\frac{m_{s,t}}{1+i_t}+a_{s,t}+m_{s,t}-m_{s,t}=\gamma\frac{a_{s,t+1}}{1+r_t}+\gamma\frac{m_{s,t+1}}{1+r_t}-\gamma\frac{\tilde{\omega}_{s,t+1}-c_{s,t+1}}{1+r_t}, \tag{15T}$$

or equivalently

$$a_{s,t}+m_{s,t}=\gamma\frac{a_{s,t+1}}{1+r_t}+\gamma\frac{m_{s,t+1}}{1+r_t}-\gamma\frac{\tilde{\omega}_{s,t+1}-c_{s,t+1}}{1+r_t}+\frac{i_t}{1+i_t}m_{s,t}. \tag{16T}$$

Recalling the budget constraint, the above equation can be re-written as

$$\gamma\tilde{\omega}_{s,t} - \gamma c_{s,t} + \frac{m_{s,t-1}}{1+\pi_t} + (1+r_{t-1})a_{s,t-1} = \gamma^2 \frac{a_{s,t+1}}{1+r_t} + \gamma^2 \frac{m_{s,t+1}}{1+r_t} +$$

$$-\gamma^2 \frac{\tilde{\omega}_{s,t+1} - c_{s,t+1}}{1+r_t} + \gamma \frac{i_t m_{s,t}}{1+i_t}. \qquad (17T)$$

Iterating the procedure, using the solvency constraint, $\lim_{t\to\infty} \gamma^t R_t (a_{s,t} + m_{s,t}) = 0,$ yields

$$\frac{m_{s,t-1}}{1+\pi_t} + (1+r_{t-1})a_{s,t-1} = -\gamma\sum_{v=0}^{\infty}\gamma^v \frac{R_{t+v}}{R_t}(w_{s,t+v} - \tau_{s,t+v}) + \gamma\sum_{v=0}^{\infty}\gamma^v \frac{R_{t+v}}{R_t} c_{s,t+v} +$$

$$+\gamma\sum_{v=0}^{\infty}\gamma^v \frac{R_{t+v}}{R_t}\frac{i_{t+v}}{1+i_{t+v}} m_{s,t+v}. \qquad (18T)$$

Rearranging

$$\sum_{v=0}^{\infty}\gamma^v \frac{R_{t+v}}{R_t}\left(c_{s,t+v} + \frac{i_{t+1}}{1+i_{t+1}} m_{s,t+v}\right) = \frac{1}{\gamma}\frac{m_{s,t-1}}{1+\pi_t} + \frac{1}{\gamma}(1+r_{t-1})a_{s,t-1} +$$

$$+\sum_{v=0}^{\infty}\gamma^v \frac{R_{t+v}}{R_t}(w_{s,t+v} - \tau_{s,t+v}) \qquad (19T)$$

Recalling conditions (11T) and (12T), equation (19T) becomes

$$\frac{1+\eta}{1-\gamma\beta} c_{s,t} = \frac{1}{\gamma}\frac{m_{s,t-1}}{1+\pi_t} + \frac{1}{\gamma}(1+r_{t-1})a_{s,t-1} + \sum_{v=0}^{\infty}\gamma^v \frac{R_{t+v}}{R_t}(w_{s,t+v} - \tau_{s,t+v}). \qquad (20T)$$

Rearranging the above equation, the closed form solution for consumption is

$$c_{s,t} = \frac{1}{\gamma}\frac{1-\gamma\beta}{1+\eta}\left[\frac{m_{s,t-1}}{1+\pi_t} + (1+r_{t-1})a_{s,t-1}\right] + \frac{1-\gamma\beta}{1+\eta}\sum_{v=0}^{\infty}\gamma^v \frac{R_{t+v}}{R_t}(w_{s,t+v} - \tau_{s,t+v}), \qquad (21T)$$

Which can be expressed in compact form as

$$c_{s,t} = \frac{1-\beta\gamma}{1+\eta}\left[\frac{1}{\gamma}\left(\frac{m_{s,t-1}}{1+\pi_t} + \frac{R_{t-1}}{R_t} a_{s,t-1}\right) + h_{s,t}\right], \qquad (22T)$$

where $h_{s,t}$ is the human wealth, defined as the present discounted value of future labor incomes net of taxes $h_{t,s} = \sum_{v=0}^{\infty}\gamma^v \frac{R_{t+v}}{R_t}(w_{s,t+v} - \tau_{s,t+v}).$

Summing up across generations and recalling that newly born agents (a fraction γ of total population) do not hold any financial wealth, give the aggregate consumption function:

$$C_t = \frac{1-\beta\gamma}{1+\eta}\left[\frac{M_{t-1}}{1+\pi_t} + (1+r_{t-1})A_{t-1} + H_t\right]. \tag{23T}$$

Derivation of the difference equation describing the time path of aggregate consumption (eq. 3)

Given the definition of human wealth, it follows

$$H_t = W_t - T_t + \frac{\gamma}{1+r_t}H_{t+1}. \tag{24T}$$

Solving for H_{t+1} and substituting into (23T) updated of one period yield

$$C_{t+1} = \frac{1-\gamma\beta}{1+\eta}\frac{1}{\gamma}(1+r_t)H_t - (1+r_t)\frac{1-\gamma\beta}{1+\eta}\frac{1}{\gamma}(W_t - T_t) +$$
$$+(1+r_t)\frac{1-\gamma\beta}{1+\eta}A_t + \frac{1-\gamma\beta}{1+\eta}\frac{M_t}{1+\pi_{t+1}}. \tag{25T}$$

From the closed form solution (23T)

$$C_{t+1} = \frac{1}{\gamma}(1+r_t)\left\{C_t - \frac{1-\beta\gamma}{1+\eta}\left[\frac{M_{t-1}}{1+\pi_t} + (1+r_{t-1})A_{t-1}\right]\right\} - (1+r_t)\frac{1-\gamma\beta}{1+\eta}\frac{1}{\gamma}(W_t - T_t) +$$
$$+(1+r_t)\frac{1-\gamma\beta}{1+\eta}A_t + \frac{1-\gamma\beta}{1+\eta}\frac{M_t}{1+\pi_{t+1}}. \tag{26T}$$

Combining the above equation with the aggregate budget constraint (5) yields

$$C_{t+1} = \frac{1}{\gamma}(1+r_t)\left[C_t - \frac{1-\beta\gamma}{1+\eta}(A_t + M_t + C_t)\right] + (1+r_t)\frac{1-\beta\gamma}{1+\eta}A_t +$$
$$+\frac{1-\beta\gamma}{1+\eta}\frac{M_t}{1+\pi_{t+1}}. \tag{27T}$$

Then consider the following manipulations:

- expand equation (27T)

$$C_{t+1} = \tfrac{1}{\gamma}(1+r_t)C_t - \tfrac{1}{\gamma}(1+r_t)\tfrac{1-\beta\gamma}{1+\eta}C_t - \tfrac{1}{\gamma}(1+r_t)\tfrac{1-\beta\gamma}{1+\eta}M_t$$
$$+\tfrac{1-\beta\gamma}{1+\eta}\tfrac{M_t}{1+\pi_{t+1}} - (1+r_t)\tfrac{1-\gamma}{\gamma}\tfrac{1-\beta\gamma}{1+\eta}A_t \ ,$$

- multiply and divide by the inflation factor the first term referring to real money balances

$$C_{t+1} = \tfrac{1}{\gamma}(1+r_t)C_t - \tfrac{1}{\gamma}(1+r_t)\tfrac{1-\beta\gamma}{1+\eta}C_t - \tfrac{1}{\gamma}\tfrac{(1+r_t)(1+\pi_{t+1})}{1+\pi_{t+1}}\tfrac{1-\beta\gamma}{1+\eta}M_t$$
$$+\tfrac{1-\beta\gamma}{1+\eta}\tfrac{M_t}{1+\pi_{t+1}} - (1+r_t)\tfrac{1-\gamma}{\gamma}\tfrac{1-\beta\gamma}{1+\eta}A_t \ ,$$

- add and subtract the term $\tfrac{1}{\gamma}\tfrac{1}{1+\pi_{t+1}}\tfrac{1-\beta\gamma}{1+\eta}M_t$

$$C_{t+1} = \tfrac{1}{\gamma}(1+r_t)C_t - \tfrac{1}{\gamma}(1+r_t)\tfrac{1-\beta\gamma}{1+\eta}C_t - \tfrac{1}{\gamma}\tfrac{1+i_t}{1+\pi_{t+1}}\tfrac{1-\beta\gamma}{1+\eta}M_t$$

$$+\tfrac{1-\beta\gamma}{1+\eta}\tfrac{M_t}{1+\pi_{t+1}} - \tfrac{1}{\gamma}\tfrac{1}{1+\pi_{t+1}}\tfrac{1-\beta\gamma}{1+\eta}M_t + \tfrac{1}{\gamma}\tfrac{1}{1+\pi_{t+1}}\tfrac{1-\beta\gamma}{1+\eta}M_t - (1+r_t)\tfrac{1-\beta\gamma}{1+\eta}A_t\tfrac{1-\gamma}{\gamma} \;,$$

- simplify

$$C_{t+1} = \tfrac{1}{\gamma}(1+r_t)C_t - \tfrac{1}{\gamma}(1+r_t)\tfrac{1-\beta\gamma}{1+\eta}C_t - \tfrac{1}{\gamma}\tfrac{(1+i_t)-1}{1+\pi_{t+1}}\tfrac{1-\beta\gamma}{1+\eta}M_t - \tfrac{1-\gamma}{\gamma}\tfrac{1-\beta\gamma}{1+\eta}\tfrac{M_t}{1+\pi_{t+1}}$$

$$-(1+r_t)(\tfrac{1-\gamma}{\gamma})\tfrac{(1-\beta\gamma)}{(1+\eta)}A_t \;,$$

- recall condition (4)

$$C_{t+1} = \tfrac{1}{\gamma}(1+r_t)C_t - \tfrac{1}{\gamma}(1+r_t)\tfrac{1-\beta\gamma}{1+\eta}C_t - \tfrac{1}{\gamma}\tfrac{\eta(1+i_t)}{1+\pi_{t+1}}\tfrac{1-\beta\gamma}{1+\eta}C_t - \tfrac{1-\gamma}{\gamma}\tfrac{1-\beta\gamma}{1+\eta}\tfrac{M_t}{1+\pi_{t+1}}$$

$$-(1+r_t)(\tfrac{1-\gamma}{\gamma})\tfrac{(1-\beta\gamma)}{(1+\eta)}A_t \;,$$

- finally note that

$$\tfrac{1}{\gamma}(1+r_t)C_t[1-\tfrac{1-\beta\gamma}{1+\eta}-\eta\tfrac{1-\beta\gamma}{1+\eta}] = \tfrac{1}{\gamma}(1+r_t)C_t[\tfrac{1+\eta-1+\beta\gamma-\eta+\eta\beta\gamma}{1+\eta}]$$

$$=\beta(1+r_t)C_t \;.$$

It follows that

$$C_{t+1} = \beta(1+r_t)C_t - \frac{1-\gamma}{\gamma}\frac{1-\beta\gamma}{1+\eta}\left[\frac{M_t}{1+\pi_{t+1}} + (1+r_t)A_t\right]. \tag{28T}$$

This shows equation (3) of the main text.

Equilibrium condition in the money market

Rewrite the portfolio balance condition (4) as follows

$$\eta C_t = \frac{(1+\pi_{t+1})(1+r_t)-1}{(1+\pi_{t+1})(1+r_t)}M_t. \tag{29T}$$

or better

$$\eta C_t(1+\pi_{t+1})(1+r_t) = (1+\pi_{t+1})(1+r_t)M_t - M_t. \tag{30T}$$

Updating equation (7) of one period and substituting into (30T) give

$$\eta C_t(1+\pi_{t+1})(1+r_t) = (1+\pi_{t+1})(1+r_t)M_t - M_{t+1}\frac{1+\pi_{t+1}}{1+\mu_{t+1}}. \tag{31T}$$

Simplifying

$$M_{t+1} = (1+\mu_{t+1})(1+r_t)(M_t - \eta C_t). \tag{32T}$$

This shows equation (13) of the main text.

Steady state conditions and implied parameter values

In steady state equation (11) can be re-written as

$$\delta\frac{K}{Y}=1-\frac{C}{Y}-\frac{G}{Y}, \tag{33T}$$

while $Y_t=\kappa F(K_{t-1},1)=(r_{t-1}+\delta)K_{t-1}+W_t$ becomes

$$1=(r+\delta)\frac{K}{Y}+(1-\alpha) \tag{34T}$$

The rate of depreciation δ and the capital/income ratio must satisfy both conditions. It follows that

$$\frac{K}{Y}=\frac{\alpha}{r}-(1-\frac{C}{Y}-\frac{G}{Y}), \tag{35T}$$

$$\delta=\frac{(1-\frac{C}{Y}-\frac{G}{Y})}{\frac{\alpha}{r}-(1-\frac{C}{Y}-\frac{G}{Y})}. \tag{36T}$$

The subjective discount factor β is given by equation (3) in steady state

$$\beta=\frac{\frac{C}{Y}(1+\eta)+\frac{1-\gamma}{\gamma}[\frac{1}{1+\pi}\frac{M}{Y}+(1+r)\frac{A}{Y}]}{(1+r)(1+\eta)\frac{C}{Y}+(1-\gamma)[\frac{1}{1+\pi}\frac{M}{Y}+(1+r)\frac{A}{Y}]}. \tag{37T}$$

The coefficient η is derived from equation (13)

$$\eta=\frac{M}{C}\frac{r+\mu(1+r)}{(1+\mu)(1+r)}. \tag{38T}$$

Finally, the critical parameter of the wealth tax must be chosen so as to keep the budget constraint of the government (12) on balance

$$\theta=r+\frac{G}{B}+\frac{Z}{B}-\frac{M}{B}\left(1-\frac{1}{1+\pi}\right). \tag{39T}$$

Model Solution

The non-linear system of difference equations (3) and (11)-(13) is log-linearized around the steady state, *{C, K, B, M}*, so as to obtain a linear system of difference equations where variables are expressed in terms of percentage deviations from the steady state.

Defining $\widehat{S}_t=[\widehat{K}_t \mid \widehat{B}_t \mid \widehat{M}_t \mid \widehat{C}_t]'$ and $\widehat{\varepsilon}_t=[\widehat{G}_t \mid \widehat{Z}_t]'$ the system can be written as

$$\Omega_S\widehat{S}_{t+1}=\Omega_S^L\widehat{S}_t+\Omega_\varepsilon\widehat{\varepsilon}_{t+1}, \tag{40T}$$

$$\widehat{\varepsilon}_{t+1}=\Sigma\widehat{\varepsilon}_t, \tag{41T}$$

where each element of vectors $\widehat{S}_t$ and $\widehat{\varepsilon}_t$ say X_t, is expressed as $\widehat{X}_t = \ln(\frac{X_t}{X}) \cong \frac{X_t - X}{X}$.

The relevant matrices of the system are defined as

$$\Omega_S = \begin{pmatrix} \frac{K}{Y} & 0 & 0 & \frac{C}{Y} \\ 0 & 1 & \frac{\mu}{1+\mu}\frac{M}{B} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1-\gamma}{\gamma}\frac{1-\gamma\beta}{1+\eta}\frac{1}{1+\mu}\frac{M}{C} & 1 \end{pmatrix}$$

$$\Omega_S^L = \begin{pmatrix} \alpha+(1-\delta)\frac{K}{Y} & 0 & 0 & 0 \\ -(1-\alpha)(r+\delta) & 1+r-\theta & 0 & 0 \\ \omega_{31} & 0 & (1+\mu)(1+r) & \omega_{34} \\ \omega_{41} & \omega_{42} & 0 & \beta(1+r) \end{pmatrix}$$

with

$\omega_{31} \equiv (1+\mu)(1-\alpha)(r+\delta)(\eta\frac{C}{M}-1)$

$\omega_{34} \equiv -(1+\mu)(1+r)\eta\frac{C}{M}$

$\omega_{41} \equiv -\frac{1-\gamma}{\gamma}\frac{1-\gamma\beta}{1+\eta}(1+r)\frac{K}{C} + \frac{1-\gamma}{\gamma}\frac{1-\gamma\beta}{1+\eta}(1-\alpha)(r+\delta)(\frac{B}{C}+\frac{K}{C}) - \beta(1-\alpha)(r+\delta)$

$\omega_{42} \equiv -\frac{(1-\gamma)(1-\beta\gamma)}{(1+\eta)\gamma}(1+r)\frac{B}{C}$

$$\Omega_\varepsilon = \begin{pmatrix} -\frac{G}{Y} & 0 \\ \frac{G}{B} & \frac{Z}{B} \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\Sigma = \begin{pmatrix} \rho_G & 0 \\ 0 & \rho_Z \end{pmatrix}.$$

Matrix Σ presents the autoregressive coefficients of the shocks and indicates how exogenous fiscal variables evolve over time.

The log-linearized model (40T)-(41T) can be solved by applying the Blanchard-Kahn (1980) algorithm where $\widehat{K}_t$ and $\widehat{B}_t$ are predetermined variables and $\widehat{M}_t$ and $\widehat{C}_t$ are forward looking variables. The solution, describing the fluctuations of the economic variables of the system around their steady state values in response to changes in public expenditure and transfers, takes the following form

$$\widehat{S}_{1,t+1} = \Theta\widehat{S}_{1,t} + \Phi\widehat{\varepsilon}_t, \qquad (42T)$$

$$\widehat{S}_{2,t} = \Upsilon\widehat{S}_{1,t} + \Psi\widehat{\varepsilon}_t, \qquad (43T)$$

where $\widehat{S}_1$ and $\widehat{S}_2$ are the subvectors of endogenous states and of forward looking variables, respectively. The elements of the matrices Θ, Φ, Υ and Ψ depend on the underlying parameters and on the critical ratios of the model.

In order to better understand the dynamic properties of the model in response to changes in $\widehat{\varepsilon}_t$, we characterize the dynamics of three other forward looking variables (the inflation rate, the price level and the nominal interest rate) and of another predetermined variable (the real interest rate).

The path for the inflation rate and for the price level as percentage deviations from their trend are

$$\tilde{\pi}_{t+1} = \hat{\pi}_{t+1}\mu = -(1+\mu)\widehat{M}_{t+1} + (1+\mu)\widehat{M}_t \tag{44T}$$

$$\widehat{P}_t = -\widehat{M}_t \tag{45T}$$

where $\tilde{\pi}_{t+1} = \pi_{t+1} - \pi$. Finally, the time path for the real and the nominal interest rate as deviations from their steady state levels are

$$\tilde{r}_t = r_t - r = (r+\delta)(\alpha-1)\widehat{K}_t \tag{46T}$$

$$\tilde{i}_t = i_t - i = (1+r)\tilde{r}_t + (1+\pi)\tilde{\pi}_{t+1} \tag{47T5}$$

The time path of the nominal interest rate turns out to be crucial in the determination of the equilibrium conditions in the money market and in the characterization of the effects produced by fiscal expansions in the simulations.

Computations are performed under the assumption that the rate of money growth is constant over time, that is $\mu_{s,t} = \mu$ for each t.

Numerical Solution

The relevant matrices that describe the model solution under the assumption of an annual government debt-GDP ratio equal to 60% are the following:

$$\Theta_{60\%} = \begin{pmatrix} .9728 & -.0002 \\ -.0206 & .8091 \end{pmatrix},$$

$$\Phi_{60\%} = \begin{pmatrix} -.0107 & -.0003 \\ .0815 & .1025 \end{pmatrix},$$

$$\Upsilon_{60\%} = \begin{pmatrix} .6949 & -.0003 \\ .7093 & .0045 \end{pmatrix},$$

$$\Psi_{60\%} = \begin{pmatrix} -.1449 & -.0024 \\ -.1292 & .0050 \end{pmatrix}.$$

In the absence of wealth effects the relevant matrices are:

$$\Theta_{NW} = \begin{pmatrix} .9732 & 0 \\ -.0206 & .8091 \end{pmatrix},$$

$$\Phi_{NW} = \begin{pmatrix} -.0101 & 0 \\ .0815 & .1025 \end{pmatrix},$$

$$\Upsilon_{NW} = \begin{pmatrix} .6997 & 0 \\ .6997 & 0 \end{pmatrix},$$

$$\Psi_{NW} = \begin{pmatrix} -.1413 & 0 \\ -.1413 & 0 \end{pmatrix}.$$

Finally, under the assumption of an annual government debt-GDP ratio equal to 106% the solution is described by the following matrices:

$$\Theta_{106\%} = \begin{pmatrix} .9732 & -.0005 \\ -.0198 & .8920 \end{pmatrix},$$

$$\Phi_{106\%} = \begin{pmatrix} -.0109 & -.0004 \\ .0461 & .0580 \end{pmatrix},$$

$$\Upsilon_{106\%} = \begin{pmatrix} .7011 & -.0018 \\ .6990 & .0119 \end{pmatrix},$$

$$\Psi_{106\%} = \begin{pmatrix} -.1483 & -.0044 \\ -.1252 & .0075 \end{pmatrix}.$$

The above matrices have been used to generate the responses of any variable in the vector $\widehat{S}$ to any of the changes in the vector $\hat{\varepsilon}$ under the assumption that autoregressive coefficients are

$$\Sigma = \begin{pmatrix} .95 & 0 \\ 0 & .95 \end{pmatrix}.$$