Population Dynamics and Monetary Policy
Technical Appendix

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Solution to the consumer's optimization problem

The representative consumer of the generation born at time s chooses the optimal sequence of consumption, $\overline{c}_{s,v}$, and real money balances, $\overline{m}_{s,v}$, in order to maximize her lifetime utility function

$$\int_t^{\infty} \log \Omega \left(\overline{c}_{s,v}, \overline{m}_{s,v} \right) e^{-(\mu+\rho)(v-t)} dv, \tag{1T}$$

subject to the budget constraint

$$\overset{\bullet}{\overline{a}}_{s,t} = \left(R_t - \pi_t + \mu \right) \overline{a}_{s,t} + \overline{y}_{s,t} - \overline{\tau}_{s,t} - \overline{c}_{s,t} - R_t \overline{m}_{s,t}, \tag{2T}$$

and to the transversality condition

$$\lim_{v \to \infty} \overline{a}_{s,v} e^{-\int_t^v (R_u - \pi_u + \mu) du} \geq 0, \tag{3T}$$

where all variables are defined as in the main text.

Let $\overline{z}_{s,t}$ denote total consumption, defined as the sum of consumption plus the interest foregone on real money holdings: $\overline{z}_{s,t} = \overline{c}_{s,t} + R_t \overline{m}_{s,t}$.

The consumer's problem can be solved by using a two-stage budgeting procedure.

First stage In the first stage, the representative consumer of generation born at time s chooses the optimal mix of consumption and real money holdings so as to maximize the instantaneous utility function, $\log \Omega \left(\overline{c}_{s,t}, \overline{m}_{s,t} \right)$, for a given level of total consumption, $\overline{z}_{s,t}$. The first order condition for a maximum is:

$$\frac{\Omega_m \left(\overline{c}_{s,t}, \overline{m}_{s,t} \right)}{\Omega_c \left(\overline{c}_{s,t}, \overline{m}_{s,t} \right)} = R_t. \tag{4T}$$

which can be re-written as

$$\overline{c}_{s,t} = \Gamma(R_t) \overline{m}_{s,t}, \tag{5T}$$

since $\Omega(\cdot, \cdot)$ is assumed to be a linearly homogenous function. Function $\Gamma(\cdot)$ is such that $\Gamma'(R_t) > 0$ and $\Gamma(R_t) - R_t \Gamma'(R_t) > 0$, where the latter condition follows from the assumption that the elasticity of substitution between real money balances and consumption, say σ, is less than one. From (5T) in fact:

$$d\frac{\overline{c}_{s,t}}{\overline{m}_{s,t}} = \Gamma'(R_t)\, dR_t. \tag{6T}$$

Dividing both sides of (6T) by $\frac{\overline{c}_{s,t}}{\overline{m}_{s,t}}$ and recalling (5T) give:

$$\frac{d\frac{c_{s,t}}{m_{s,t}}}{\frac{c_{s,t}}{m_{s,t}}} = \frac{\Gamma'(R_t)}{\Gamma(R_t)} R_t \frac{dR_t}{R_t}. \tag{7T}$$

It follows that the elasticity of substitution between real money balances and consumption can be expressed as:

$$\sigma = \frac{d\frac{c_{s,t}}{m_{s,t}}}{\frac{c_{s,t}}{m_{s,t}}} / \frac{dR_t}{R_t} = \frac{\Gamma'(R_t)}{\Gamma(R_t)} R_t. \tag{8T}$$

Second stage In the second stage, the representative consumer born at time s solves an intertemporal optimization problem and derives the optimal time path of total consumption, $\overline{z}_{s,t}$. Using the definition of total consumption and the optimal condition (5T), the instantaneous utility function can be expressed in function of total consumption:

$$\log \Omega(\overline{c}_{s,t}, \overline{m}_{s,t}) = \log \Omega\left(\frac{\Gamma(R_t)}{\Gamma(R_t) + R_t} \overline{z}_{s,t}, \frac{1}{\Gamma(R_t) + R_t} \overline{z}_{s,t}\right). \tag{9T}$$

Since $\Omega(\overline{c}_{s,t}, \overline{m}_{s,t})$ is linearly homogenous, the instantaneous utility function can be written in a more compact fashion as:

$$\log \Omega(\overline{c}_{s,t}, \overline{m}_{s,t}) = \log q_t + \log \overline{z}_{s,t}, \tag{10T}$$

where $q_t \equiv \Omega\left(\frac{\Gamma(R_t)}{\Gamma(R_t)+R_t}, \frac{1}{\Gamma(R_t)+R_t}\right)$.

Using this result, the intertemporal optimization problem can be stated as follows:

$$\max_{\{\overline{z}_{s,t}\}} \int_t^\infty [\log q_t + \log \overline{z}_{t,v}]\, e^{-(\mu+\rho)t} dt, \tag{11T}$$

subject to

$$\dot{\overline{a}}_{s,t} = (R_t - \pi_t + \mu)\, \overline{a}_{s,t} + \overline{y}_{s,t} - \overline{\tau}_{s,t} - \overline{z}_{s,t}, \tag{12T}$$

and given $\overline{a}_{s,t}$ and for $\overline{z} > 0$.

The optimal solution is obtained by setting up the following current-value Hamiltonian function:

$$H_t = \left\{[\log q_t + \log \overline{z}_{s,t}] + \overline{\lambda}_{s,t}\left[(R_t - \pi_t + \mu)\, \overline{a}_{s,t} + \overline{y}_{s,t} - \overline{\tau}_{s,t} - \overline{z}_{s,t}\right]\right\} e^{-(\mu+\rho)t},$$

where $\overline{a}_{s,t}$ is the state variable, $\overline{z}_{s,t}$ is the control variable and $\overline{\lambda}_{s,t}$ is the co-state variable.

Necessary and sufficient conditions for the optimum are:
$\frac{\partial H_t}{\partial \overline{z}_{s,t}} = 0 \rightarrow \overline{z}_{s,t} = \frac{1}{\overline{\lambda}_{s,t}}$,

$\frac{d\overline{\lambda}_{s,t}e^{-(\mu+\rho)t}}{dt} = -\frac{\partial H_t}{\partial \overline{a}_{s,t}} \rightarrow \overset{\bullet}{\overline{\lambda}}_{s,t} = (\rho - R_t + \pi_t)\overline{\lambda}_{s,t}$,

Combining the above conditions we obtain the optimal path of total consumption for an individual:

$$\overset{\bullet}{\overline{z}}_{s,t} = (R_t - \pi_t - \rho)\overline{z}_{s,t}. \tag{13T}$$

This shows equation (8) of the main text.

Derivation of equation (9)

A closed form solution for total consumption at the individual level can be derived as follows. Consider the instantaneous budget constraint:

$$\overset{\bullet}{\overline{a}}_{s,t} - (R_t - \pi_t + \mu)\,\overline{a}_{s,t} = \overline{y}_{s,t} - \overline{\tau}_{s,t} - \overline{z}_{s,t}. \tag{14T}$$

Multiplying both terms by factor $e^{-\int_t^v (R_u - \pi_u + \mu)du}$ and integrating them forward, (14T) becomes:

$$\begin{aligned} & \int_t^\infty \left[\overset{\bullet}{\overline{a}}_{s,v} - (R_v - \pi_v + \mu)\,\overline{a}_{s,v}\right] e^{-\int_t^v (R_u - \pi_u + \mu)du} dv \\ = & \int_t^\infty \left(\overline{y}_{s,v} - \overline{\tau}_{s,v} - \overline{z}_{s,v}\right) e^{-\int_t^v (R_u - \pi_u + \mu)du} dv. \end{aligned} \tag{15T}$$

The LHS of the above equation can be simplified as follows:

$$\begin{aligned} LHS &= \left[\overline{a}_{s,v} e^{-\int_t^v (R_u - \pi_u + \mu)du}\right]_t^\infty, \\ &= \lim_{v\to\infty} \overline{a}_{s,v} e^{-\int_t^v (R_u - \pi_u + \mu)du} - \overline{a}_{s,t}, \\ &= -\overline{a}_{s,t}, \end{aligned}$$

where we have used the fact that $\lim_{v\to\infty} \overline{a}_{s,v} e^{-\int_t^v (R_u - \pi_u + \mu)du} = 0$.

The RHS can be expressed as:

$$RHS = \overline{h}_{s,t} - \int_t^\infty \overline{z}_{s,v} e^{-\int_t^v (R_u - \pi_u + \mu)du} dv,$$

where we have used the definition of human wealth: $\overline{h}_{s,t} = \int_t^\infty \left(\overline{y}_{s,v} - \overline{\tau}_{s,v}\right) e^{-\int_t^v (R_u - \pi_u + \mu)du} dv$.

It follows that (15T) can be written as:

$$\int_t^\infty \overline{z}_{s,v} e^{-\int_t^v (R_u - \pi_u + \mu)du} dv = \overline{h}_{s,t} + \overline{a}_{s,t}, \tag{16T}$$

which is the intertemporal budget constraint of the individual born at time s according to which the present discounted value of total consumption must be equal to the level of total wealth (human and financial wealth).

Using condition (8), the level of total consumption at time $v \geq t$, $\overline{z}_{s,v}$, can be expressed in function of $\overline{z}_{s,t}$:

$$\overline{z}_{s,v} = \overline{z}_{s,t} e^{\int_t^v (R_u - \pi_u - \rho) du}. \tag{17T}$$

Plugging the above solution into (16T) and solving the integral give:

$$\overline{z}_{s,t} = (\mu + \rho) \left(\overline{a}_{s,t} + \overline{h}_{s,t} \right). \tag{18T}$$

This demonstrates equation (9).

Derivation of equations (10) and (11)

Combing (4) and (6) yields

$$\overline{z}_{s,t} = \left(1 + \frac{R_t}{\Gamma(R_t)} \right) \overline{c}_{s,t}, \tag{19T}$$

which can be re-written more compactly as

$$\overline{z}_{s,t} = L(R_t) \overline{c}_{s,t}, \tag{20T}$$

where $L(R_t) = 1 + \frac{R_t}{\Gamma(R_t)}$ with $L'(R_t) = \frac{\Gamma(R_t) - R_t \Gamma'(R_t)}{[\Gamma(R_t)]^2}$.

Time-differentiating (20T) gives

$$\dot{\overline{z}}_{s,t} = L'(R_t) \dot{R}_t . \overline{c}_{s,t} + \dot{\overline{c}}_{s,t} L(R_t). \tag{21T}$$

Using the above results it is straightforward to obtain the following equations for individual consumption:

$$\dot{\overline{c}}_{s,t} = (R_t - \pi_t - \rho) \overline{c}_{s,t} - \frac{L'(R_t) \dot{R}_t}{L(R_t)} \overline{c}_{s,t}, \tag{22T}$$

$$\overline{c}_{s,t} = \frac{\mu + \rho}{L(R_t)} \left(\overline{a}_{s,t} + \overline{h}_{s,t} \right). \tag{23T}$$

Derivation of equation (17)

By definition, per capita aggregate financial wealth is

$$a_t = \beta \int_{-\infty}^{t} \overline{a}_{s,t} e^{\beta(s-t)} ds. \tag{24T}$$

Differentiating with respect to time the above equation using Leibnitz's rule

$$\dot{a}_t = \beta \overline{a}_{t,t} - \beta a_t + \beta \int_{-\infty}^{t} \dot{\overline{a}}_{s,t} e^{\beta(s-t)} ds, \tag{25T}$$

where $\beta\overline{a}_{t,t}$ is the newborns' financial wealth and is equal to zero by assumption (agents are born with zero wealth since there is no bequest motive). Using (**??**), (25T) can be re-written as:

$$\begin{aligned} \overset{\bullet}{a}_t &= -\beta a_t + \mu a_t + (R_t - \pi_t)\, a_t + y_t - \tau_t - c_t - R_t m_t \\ &= (R_t - \pi_t - n)\, a_t + y_t - \tau_t - c_t - R_t m_t. \end{aligned} \tag{26T}$$

This shows (17).

Fiscal solvency

Consider equation (22) of the text:

$$\overset{\bullet}{a}_t = [R(\pi_t) - \pi_t - n - \theta]\, a_t, \tag{27T}$$

whose backward solution, given the initial condition a_0, is

$$a_t = a_0 e^{\int_0^t (R_u - \pi_u - n - \theta) du}, \tag{28T}$$

which is stable for $t \to \infty$ as long as $n + \theta \geq R_u - \pi_u$. In fact for an initial finite level of a, $\lim\limits_{t\to\infty} a_t$ is finite if $\theta + n \geq R_u - \pi_u$. Re-writing (28T) gives:

$$a_t e^{-\int_0^t (R_u - \pi_u) du} = a_0 e^{-(n+\theta)t} \tag{29T}$$

where the limit for $t \to \infty$ of the RHS is zero, given any initial finite value of a as long as $n + \theta > 0$. The limit of the LHS is also equal to zero as long as the limit of a_t is finite and this is the case since equation (27T) is a stable process. It should be noted that at the *intended* steady state $R_u - \pi_u = \rho$, hence $\rho - n - \theta < 0$ is a sufficient condition for fiscal solvency in its neighborhood.

Steady State Equilibria

The steady-state equilibrium is defined from equations (21) and (22) setting $\overset{\bullet}{\pi}_t, \overset{\bullet}{a}_t = 0$:

$$[R(\pi) - \pi - \rho]\, \frac{L\,[R(\pi)]}{L'\,[R(\pi)]\, R'(\pi)} - \beta(\rho + \mu) \frac{a}{L'\,[R(\pi)]\, R'(\pi)} = 0, \tag{30T}$$

$$[R(\pi) - \pi - n - \theta]\, a = 0. \tag{31T}$$

Under general conditions there exist at least two steady state equilibria (π^*, a^*) and $(\widehat{\pi}, \widehat{a})$ characterized as follows:

$$R(\pi^*) - \pi^* = \rho, \;\; a^* = 0 \tag{32T}$$

$$R(\widehat{\pi}) - \widehat{\pi} = n + \theta, \;\; \widehat{a} = \frac{[R(\widehat{\pi}) - \widehat{\pi} - \rho]\, L\,[R(\widehat{\pi})]}{\beta(\rho + \mu)} \tag{33T}$$

Local Analysis

From equations (21) and (22):

$$\frac{\partial \overset{\bullet}{\pi}_t}{\partial \pi_t} = \frac{\left[R'(\pi_t)-1\right]L[R(\pi_t)]+[R(\pi_t)-\pi_t-\rho]L'[R(\pi_t)]R'(\pi_t)}{L'[R(\pi_t)]R'(\pi_t)}$$
$$-\frac{\{L''[R(\pi_t)]R'(\pi_t)^2+L'[R(\pi_t)]R''(\pi_t)\}\{[R(\pi_t)-\pi_t-\rho]L[R(\pi_t)]-(\rho+\mu)\beta a_t\}}{\{L'[R(\pi_t)]R'(\pi_t)\}^2}$$
$$\frac{\partial \overset{\bullet}{\pi}_t}{\partial a_t} = -\frac{\beta(\rho+\mu)}{L'[R(\pi_t)]R'(\pi_t)}$$
$$\frac{\partial \overset{\bullet}{a}_t}{\partial a_t} = R(\pi_t) - \pi_t - n - \theta$$
$$\frac{\partial \overset{\bullet}{a}_t}{\partial \pi_t} = [R'(\pi_t) - 1]\, a_t$$

Using (32T) the above partial derivatives at (π^*, a^*) are equal to:

$$\begin{aligned}
\frac{\partial \overset{\bullet}{\pi}_t}{\partial \pi_t} &= [R'(\pi^*) - 1] \frac{L\,[R(\pi^*)]}{L'\,[R(\pi^*)]\,R'(\pi^*)}, \\
\frac{\partial \overset{\bullet}{\pi}_t}{\partial a_t} &= -\frac{\beta(\rho+\mu)}{L'\,[R(\pi^*)]\,R'(\pi^*)}, \\
\frac{\partial \overset{\bullet}{a}_t}{\partial a_t} &= \rho - n - \theta, \\
\frac{\partial \overset{\bullet}{a}_t}{\partial \pi_t} &= 0.
\end{aligned}$$

Using (33T) the corresponding partial derivatives at $(\widehat{\pi}, \widehat{a})$ are equal to:

$$\begin{aligned}
\frac{\partial \overset{\bullet}{\pi}_t}{\partial \pi_t} &= \frac{[R'(\widehat{\pi}) - 1]\, L\,[R(\widehat{\pi})]}{L'\,[R(\widehat{\pi})]\, R'(\widehat{\pi})} + [R(\widehat{\pi}) - \widehat{\pi} - \rho]\,, \\
\frac{\partial \overset{\bullet}{\pi}_t}{\partial a_t} &= -\frac{\beta(\rho+\mu)}{L'\,[R(\widehat{\pi})]\, R'(\widehat{\pi})}, \\
\frac{\partial \overset{\bullet}{a}_t}{\partial a_t} &= 0, \\
\frac{\partial \overset{\bullet}{a}_t}{\partial \pi_t} &= [R'(\widehat{\pi}) - 1] \frac{[R(\widehat{\pi}) - \widehat{\pi} - \rho]\, L\,[R(\widehat{\pi})]}{\beta(\rho+\mu)}.
\end{aligned}$$