Taylor Rules, Long-Run Growth and Real Uncertainty
Appendix*

Barbara Annicchiarico[†] Lorenza Rossi[‡]

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1 Intermediate goods-producing firms

The production function of the typical intermediate-good producer j is

$$Y_{j,t} = A_t \left(K_{j,t}\right)^{1-\alpha} \left(Z_t N_{j,t}\right)^{\alpha}. \tag{A-1}$$

Cost minimization, taking the nominal wage rate W_t and the rental cost of capital R_t^K as given, yields the standard optimality conditions,

$$\frac{W_t}{P_t} = \alpha MC_{j,t} \frac{Y_{j,t}}{N_{j,t}}, \tag{A-2}$$

and

$$\frac{R_t^K}{P_t} = (1-\alpha) MC_{j,t} \frac{Y_{j,t}}{K_{j,t}}, \tag{A-3}$$

thus, real marginal cost, MC_t, obtained combining the two equations, is common to all firms and given by:

$$MC_t = \frac{1}{\alpha^{\alpha}\left(1-\alpha\right)^{1-\alpha}} \frac{1}{A_t Z_t^{\alpha}} \left(\frac{R_t^K}{P_t}\right)^{1-\alpha} \left(\frac{W_t}{P_t}\right)^{\alpha}. \tag{A-4}$$

The typical firm j able to set its price optimally at time t maximizes the present discounted value of expected real profits generated while the price remains unchanged. Let $Q_{t,t+i}$ denote the stochastic discount factor, formally, the Lagrangian function is

$$E_t \sum_{i=0}^{\infty} \xi_p^i Q_{t,t+i} \left\{ \begin{array}{c} \frac{P_{j,t}^*}{P_{t+i}} Y_{j,t+i} - \frac{W_{t+i}}{P_{t+i}} L_{j,t+i} - \frac{R_{K,t}}{P_{t+i}} K_{j,t+i} + \\ + MC_{j,t+i}\left(A_t K_{j,t}^{1-\alpha}\left(Z_t N_{j,t}\right)^{\alpha} - Y_{j,t+i}\right) \end{array} \right\}, \tag{A-5}$$

*This online Appendix is not self-contained and assumes that readers are familiar with the notation used in the main text.

[†]Department of Economics, University of Rome 'Tor Vergata'.
[‡]Department of Economics and Management, University of Pavia. E-mail address: lorenza.rossi@eco.unipv.it

given the demand schedule $Y_{j,t+i} = \left(\frac{P^*_{j,t}}{P_{t+i}}\right)^{-\theta} Y_{t+i}$. The first-order condition associated to the above problem gives

$$\frac{P^*_t}{P_t} = \frac{\theta_p}{\theta_p - 1} \frac{E_t \sum_{i=0}^{\infty} \xi_p^i Q_{t,t+i} MC_{t+i} \left(\frac{P_{t+i}}{P_t}\right)^{\theta_p} Y_{t+i}}{E_t \sum_{i=0}^{\infty} \xi_p^i Q_{t,t+i} \left(\frac{P_{t+i}}{P_t}\right)^{\theta_p - 1} Y_{t+i}}, \tag{A-6}$$

where we have dropped the j-index since all firms able to set their price optimally at time t will make the same decisions. Now define two artificial variables, namely, $x_t = E_t \sum_{i=0}^{\infty} \xi_p^i Q_{t,t+i} MC_{t+i} \left(\frac{P_{t+i}}{P_t}\right)^{\theta_p} Y_{t+i}$ and $z_t = E_t \sum_{i=0}^{\infty} \xi_p^i Q_{t,t+i} \left(\frac{P_{t+i}}{P_t}\right)^{\theta_p - 1} Y_{t+i}$ and denote $p^*_t = \frac{P^*_t}{P_t}$.

Notice that the optimal price condition for the typical firm able to reset its price at time t can be re-written as

$$P^*_t = \frac{\theta_p}{\theta_p - 1} \frac{E_t \sum_{i=0}^{\infty} \xi_p^i Q_{t,t+i} MC_{t+i} \left(\frac{P_{t+i}}{P_t}\right)^{\theta_p} Y_{t+i}}{E_t \sum_{i=0}^{\infty} \xi_p^i Q_{t,t+i} \left(\frac{P_{t+i}}{P_t}\right)^{\theta_p - 1} Y_{t+i}} \frac{1}{MC_t} MC^N_t, \tag{A-7}$$

where the term $\frac{\theta_p}{\theta_p - 1} \frac{E_t \sum_{i=0}^{\infty} \xi_p^i Q_{t,t+i} MC_{t+i} \left(\frac{P_{t+i}}{P_t}\right)^{\theta_p} Y_{t+i}}{E_t \sum_{i=0}^{\infty} \xi_p^i Q_{t,t+i} \left(\frac{P_{t+i}}{P_t}\right)^{\theta_p - 1} Y_{t+i}} \frac{1}{MC_t}$ is the optimal markup of the re-optimizing firm and MC^N_t is the nominal marginal cost. Clearly, this markup depends positively on the expected discounted value of real marginal costs stemming from a price change occurring at time t. Diminishing marginal productivity of labor implies that the marginal cost is a convex function of labor inputs. For this reason, a higher variability in labor inputs, due to real uncertainty, tends to increase the expected future marginal costs, thus increasing the price set by firms and implying a higher markup.

Notice that in the limiting of flexible prices (i.e. with $\xi_p = 0$), the above condition collapses to the familiar first-order condition:

$$P^*_t = P_t = \frac{\theta_p}{\theta_p - 1} MC^N_t. \tag{A-8}$$

Of course, in this case, monetary neutrality is restored, in the sense that monetary policy does not influence nominal prices that instead adjust instantaneously to meet any change in the nominal marginal cost so as to keep the markup at the optimal level, namely $\frac{\theta_p}{\theta_p - 1}$. As a consequence, the markup channel through which uncertainty affects growth is not operative.

Using the fact that is the stochastic discount factor $Q_{t,t+i}$ used at time t by shareholders to value date $t+i$, is defined as $Q_{t,t+i} = \beta^i \frac{C_t P_t}{C_{t+i} P_{t+i}}$, the optimal price equation (A-6) can be re-written as follows

$$p^*_t = \frac{\theta_p}{\theta_p - 1} \frac{x_t}{z_t}, \tag{A-9}$$

where x_t can be written recursively as

$$x_t = C_t^{-1} Y_t MC_t + \xi_p \beta E_t \pi_{t+1}^{\theta_p} x_{t+1}, \tag{A-10}$$

while z_t can be written as

$$z_t = C_t^{-1} Y_t + \xi_p \beta E_t \pi_{t+1}^{\theta_p - 1} z_{t+1}, \tag{A-11}$$

where $\pi_t = P_t / P_{t-1}$.

Finally, the aggregate price level $P_t = \left(\int_0^1 P_{j,t}^{1-\theta_p} dj\right)^{1/(1-\theta_p)}$ evolves according to

$$P_t = \left[\xi_p P_{t-1}^{1-\theta_p} + \left(1-\xi_p\right) P_t^{*1-\theta_p}\right]^{1/(1-\theta_p)}, \tag{A-12}$$

that is the price level is just a weighted average of the last period's price level and the price set by firms adjusting in the current period. This equation can be rewritten as follows:

$$1 = \xi_p \pi_t^{\theta_p - 1} + \left(1-\xi_p\right)\left(p_t^*\right)^{1-\theta_p}. \tag{A-13}$$

2 Households

The representative household chooses $C_t,\ B_t,\ N_t, I_t$ and K_{t+1} so as to maximize the following lifetime utility subject to a sequence of flow budget constraints:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(\log C_t - \mu_n \frac{N_t{}^{1+\phi}}{1+\phi}\right),\ \phi,\ \mu_n > 0 \text{ and } 0 < \beta < 1, \tag{A-14}$$

$$P_t C_t + R_t^{-1} B_{t+1} = B_t + W_t N_t + D_t + R_t^K K_t - P_t I_t - T_t, \tag{A-15}$$

where physical capital accumulates according to

$$K_{t+1} = (1-\delta) K_t + \mu_t \left(1 - S\left(\frac{I_t}{I_{t-1}}\right)\right) I_t, \tag{A-16}$$

with $S\left(\frac{I_t}{I_{t-1}}\right) = \frac{\gamma_I}{2}\left(\frac{I_t}{I_{t-1}} - g_K\right)^2$.

The first-order conditions with respect to $C_t,\ B_t,\ N_t, I_t$ and K_{t+1} that solve the consumer's problem can be written as

$$C_t^{-1} = \lambda_t, \tag{A-17}$$

$$\frac{1}{R_t} = \beta_t E_t \frac{P_t}{P_{t+1}} \frac{\lambda_{t+1}}{\lambda_t}, \tag{A-18}$$

$$\mu_n N_t{}^{\phi} = \lambda_t \frac{W_t}{P_t}, \tag{A-19}$$

$$\begin{aligned} -\lambda_t + \zeta_t \mu_t - \zeta_t \mu_t \frac{\gamma_I}{2}\left(\frac{I_t}{I_{t-1}} - g_K\right)^2 - \zeta_t \mu_t \gamma_I \left(\frac{I_t}{I_{t-1}} - g_K\right) \frac{I_t}{I_{t-1}} + \\ + \beta E_t \zeta_{t+1} \mu_{t+1} \gamma_I \left(\frac{I_{t+1}}{I_t} - g_K\right) \frac{I_{t+1}^2}{I_t^2} = 0, \end{aligned} \tag{A-20}$$

$$\beta E_t \lambda_{t+1} \tilde{R}_{t+1}^k + \beta E_t \lambda_{t+1} \zeta_{t+1} (1-\delta) - \zeta_t = 0, \tag{A-21}$$

where λ_t denotes the Lagrangian multiplier associated to the flow budget constraint expressed in real terms (A-15), ζ_t is the Lagrangian multiplier associated to the accumulation of physical capital (A-16) and $\tilde{R}_{t+1}^k = R_t^K / P_t$.

Let define Tobin's q as $q_t = \frac{\zeta_t}{\lambda_t}$, which measures the relative marginal value of installed capital with respect to consumption, hence (A-20) and (A-21) can be expressed as

$$1 = q_t \mu_t \left[1 - \frac{\gamma_I}{2} \left(\frac{I_t}{I_{t-1}} - g_K \right)^2 - \gamma_I \left(\frac{I_t}{I_{t-1}} - g_K \right) \frac{I_t}{I_{t-1}} \right] + \tag{A-22}$$
$$+ \beta E_t q_{t+1} \frac{\lambda_{t+1}}{\lambda_t} \mu_{t+1} \gamma_I \left(\frac{I_{t+1}}{I_t} - g_K \right) \frac{I_{t+1}^2}{I_t^2},$$

$$q_t = \beta E_t \frac{\lambda_{t+1}}{\lambda_t} \left[\tilde{R}_{t+1}^k + q_{t+1}(1-\delta) \right]. \tag{A-23}$$

3 Market Clearing

In equilibrium factor and good markets clear, hence the following conditions are satisfied for all t: $N_t = \int_0^1 N_{j,t} dj$, $K_t = \int_0^1 K_{j,t} dj$ and $Y_t = D_{p,t} \int_0^1 Y_{j,t}$, where using (A-1) aggregate production is found to be:

$$Y_t = A_t K_t N_t^{\alpha} \left(D_{p,t} \right)^{-1}, \tag{A-24}$$

where it is easy to see that $D_{p,t}$ is a measure of price dispersion, $D_{p,t} = \int_0^1 \left(\frac{P_{j,t}}{P_t} \right)^{-\theta_p} dj$, evolving according to a non-linear first-order difference equation:

$$D_{p,t} = \left(1 - \xi_p \right) p_t^{*-\theta_p} + \xi_p \pi_t^{\theta_p} D_{p,t-1}. \tag{A-25}$$

The resource constraint is

$$Y_t = C_t + I_t + G_t. \tag{A-26}$$

4 Stationary Model

In this economy a number of variables, such as output, consumption etc. will not be stationary along the balanced-growth path. We therefore perform a change of variables, so as to obtain a set of equilibrium conditions that involve only stationary variables. We note that non stationary variables at time t are cointegrated with K_t, while the same variables at time $t+1$ are cointegrated with K_{t+1}. We divide variables by the appropriate cointegrating factor and denote the corresponding stationary variables with lowercase letters. Equations (A-9), (A-13), (A-25), are already expressed in terms of stationary variables.

Labor and capital demands (A-2) and (A-3) are now expressed as

$$\tilde{R}_t^K = (1-\alpha) MC_t y_t, \tag{A-27}$$

$$w_t = \alpha MC_t \frac{y_t}{N_t}, \tag{A-28}$$

where $y = Y/K$ and $w = W/(PK)$. In terms of stationary variables the price related equations (A-10) and (A-11) are

$$x_t = c_t^{-1} y_t MC_t + \xi_p \beta E_t \pi_{t+1}^{\theta} x_{t+1}, \tag{A-29}$$

$$z_t = c_t^{-1} y_t + \xi_p \beta E_t \pi_{t+1}^{\theta-1} z_{t+1}. \tag{A-30}$$

The Euler equation (A-18), given (A-17), can be expressed as

$$c_t^{-1} = \beta E_t R_t \left(c_{t+1} g_{k,t+1}\right)^{-1} \frac{1}{\pi_{t+1}}, \tag{A-31}$$

where $c = C/K$ and $g_{k,t+1} = K_{t+1}/K_t$.

The labor supply (A-19) can be written as

$$w_t = \mu_n c_t N_t^{\phi}. \tag{A-32}$$

Conditions (A-22) and (A-23) become

$$c_t^{-1} q_t = \beta E_t \left(c_{t+1} g_{k,t+1}\right)^{-1} \left[\tilde{R}_{t+1}^{k} + q_{t+1}(1-\delta)\right], \tag{A-33}$$

$$\begin{aligned} 1 = {} & q_t \mu_t \left[1 - \frac{\gamma_I}{2}\left(\frac{i_t}{i_{t-1}} g_{k,t} - g_k\right)^2 - \gamma_I \left(\frac{i_t}{i_{t-1}} g_{k,t} - g_k\right)\frac{i_t}{i_{t-1}} g_{k,t}\right] + \\ & + \beta E_t q_{t+1} \left(c_{t+1} g_{k,t+1}\right)^{-1} c_t \mu_{t+1} \gamma_I \left(\frac{i_{t+1}}{i_t} g_{k,t+1} - g_k\right)\left(\frac{i_{t+1}}{i_t} g_{k,t+1}\right)^2. \end{aligned} \tag{A-34}$$

The capital accumulation equation (A-16) becomes

$$g_{k,t+1} = (1-\delta) + \mu_t \left[1 - \frac{\gamma_I}{2}\left(\frac{i_t}{i_{t-1}} g_{k,t} - g_k\right)^2\right] i_t.$$

The production function (A-24) is simply

$$y_t = A_t N_t^{\alpha} \left(D_{p,t}\right)^{-1}. \tag{A-35}$$

Finally, the resource constraint of the economy (A-26) in stationary terms is

$$y_t = c_t + i_t + g_t, \tag{A-36}$$

where $g_t = G_t/K_t$.

The competitive equilibrium of the economy under study can now be formally defined.

Definition: *A stationary competitive equilibrium is a sequence of allocations and prices* $\{c_t,\ i_t,\ g_{k,t+1},\ N_t,\ y_t,\ \tilde{R}_t^K,\ MC_t,\ w_t,\ \pi_t,\ D_{p,t},\ p_t^*,\ x_t,\ z_t\}_{t=0}^{\infty}$ *that remain bounded in some neighborhood around the deterministic steady state and satisfy equations (A-9), (A-13), (A-25), (A-27)-(A-36), given a sequence of nominal interest rate* $\{R_t\}_{t=0}^{\infty}$*, initial value for* $\{D_{p,t-1}, g_{K,t}\}$ *and a set of exogenous stochastic processes* $\{A_t, \mu_t, g_t\}_{t=0}^{\infty}$*.*

5 Model Solution

The stationarized model is solved by using a 'pure' perturbation method which amounts to a second-order Taylor approximation of the model around its deterministic steady state.[1] Notably, second-order approximations are more accurate and allow us to account for the effects of uncertainty.[2]

To study the effects of uncertainty on long-run growth (and inflation) under different monetary policy rules, we have used the theoretical means of output growth produced by the model for various sources of uncertainty. Since starting from a solution at second-order approximation, there is no closed-form solution for unconditional moments, as common practice, we report a second order approximation of these moments.[3]

6 Inflation and Growth at Business Cycle Frequencies

In Table A-1 we report the correlation between inflation and output growth at cyclical frequencies for the four interest rate rules considered in the paper. Clearly, the model displays a positive correlation between these two variables accounting for the fact that, in the absence of cost-push shock, inflation tends to increase in booms and fall in recessions.

Table A-1: Correlation between Inflation and Output Growth under All Sources of Uncertainty

SIT	WIT	TR	TRS
0.4833	0.4400	0.4156	0.6574

[1] See Judd (1998) and Schmitt-Grohé and Uribe (2004).

[2] The model has been solved in Dynare. For details, see http://www.cepremap.cnrs.fr/dynare/ and Adjemian et al. (2010).

[3] Dynare calculates theoretical moments for all the endogenous variables using the approximation method of Kim et al. (2008).

References

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